Holomorphic Poisson structures
Part II of IV: Foliations of Poisson manifolds

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Poisson bracket \( \{ - , - \} \) on \( O_X \) \iff \[
\left\{ \begin{array}{l}
\pi \in H^0(\wedge^2 T_X) \\
0 = [\pi, \pi] \in H^0(\wedge^3 T_X)
\end{array} \right.
\]

coordinates \( \mathbf{x} \)
\[
\pi = \sum_{i < j} \pi^{ij} \partial_{x_i} \wedge \partial_{x_j}
\]
\[
\pi^{ij} = \{ x^i , x^j \} \in O_X
\]

\[
\{ f , g \} = \sum_{i < j} \pi^{ij} \cdot \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial x^j} \right)
\]

**Theorem (Weinstein splitting)**

*Every point \( x \in (X, \pi) \) has a neighbourhood isomorphic to the product of a nondegenerate (i.e. symplectic) structure and a Poisson structure that vanishes at a point:*

\[
\pi = \sum_{i=1}^{r} \partial_q i \wedge \partial_p i + \tilde{\pi}(z)
\]
\[
\tilde{\pi}|_x = 0
\]
A foliation

Recall: anchor map $\pi^\# : \Omega^1_X \to \mathcal{T}_X$ sending $\alpha \mapsto \iota_\alpha \pi$.

Every $f \in \mathcal{O}_X$ has Hamiltonian vector field $\xi_f = \{f, -\} = \pi^\#(df)$

Equivalence relation on $X$:

\[ x \sim x' \iff x, x' \text{ joined by a sequence of Hamiltonian flows} \]

Equivalence classes:

\[ X = \bigsqcup_j L_j \]

Claim: $L_j$ are immersed analytic submanifolds, giving possibly singular foliation of $X$

\[ \mathcal{F}_\pi := \mathcal{O}_X \cdot \{ \xi_f \mid f \in \mathcal{O}_X \} \subset \mathcal{T}_X \]

\[ = \text{img}(\pi^\# : \Omega^1_X \to \mathcal{T}_X) \]

Jacobi identity $\iff [\xi_f, \xi_g] = \xi_{\{f, g\}} \implies [\mathcal{F}_\pi, \mathcal{F}_\pi] \subset \mathcal{F}_\pi$
Symplectic leaves

\( L \subset X \) leaf of \( \mathcal{F}_\pi \) \implies \mathcal{T}_L = \text{img}(\pi^\#|_L) \subset \mathcal{T}_X|_L \\
\implies \pi|_L \in \wedge^2 \mathcal{T}_L \subset \wedge^2 \mathcal{T}_X|_L \text{ nondegenerate} \\
\implies \omega_L := \pi|_L^{-1} \in H^0(\Omega^2_L) \text{ symplectic}

**Theorem**

If \((X, \pi)\) is any Poisson variety, then every leaf of \( \mathcal{F}_\pi \) carries a canonical holomorphic symplectic form.

**Definition**

\( \mathcal{F}_\pi \) is called the **symplectic foliation** of \((X, \pi)\).
Examples

Example (Trivial)

\[ \pi = 0 \quad \Rightarrow \quad \mathcal{F}_\pi = 0 \quad \Rightarrow \quad \text{every } p \in X \text{ is a symplectic leaf with } \omega_p = 0 \]

Example (Nondegenerate)

\[ \pi \text{ nondegenerate} \quad \Rightarrow \quad \mathcal{F}_\pi = \mathcal{T}_X \quad \Rightarrow \quad \text{leaves are the connected components of } X, \text{ symplectic form } \omega = \pi^{-1} \]

Example (Constant)

\[ X = \mathbb{C}^n_{p_i,q_i,z_j} \quad \pi = \sum \partial q_i \wedge \partial p_i \]

\[ \mathcal{F}_\pi = \text{span}\{\partial p_i, \partial q_i\} \]

Sympelctic leaves:

\[ L = \mathbb{C}^{2r} \times \{z = \text{const}\} \quad \omega_L = \sum_i dp_i \wedge dq^i |_L \]
Example: surfaces

\((X, \pi)\) smooth connected surface:

\[ \pi \in H^0(\wedge^2 T_X) = H^0(K_X^{-1}) \]

Anticanonical divisor:

\[ Y = Zeros(\pi) \subset X \]

Symplectic leaves:

- dimension two: \(X \setminus Y\) with \(\omega = \pi|_{X \setminus Y}^{-1}\)
- dimension zero: points of \(Y\) with \(\omega = 0\)
Example: Lie algebras

\[ \mathfrak{g} = \text{Lie}(G), \quad X = \mathfrak{g}^\vee, \quad f \in \mathfrak{g} \subset \mathcal{O}(X) \]

\[ \{ f, - \} : \mathcal{O}(X) \rightarrow \mathcal{O}(X) \quad \text{extends} \quad ad_f = [f, -] : \mathfrak{g} \rightarrow \mathfrak{g} \]

**Theorem (Kirillov–Kostant–Soriau)**

*Symplectic leaves of \( \mathfrak{g}^\vee \) are the orbits of the coadjoint action \( G \circlearrowright \mathfrak{g}^\vee \). In particular, each coadjoint orbit carries a canonical symplectic form.*

**NB:** \( \{0\} \subset \mathfrak{g}^\vee \) always a leaf

**Example (\( \mathfrak{g} = \mathfrak{sl}(2; \mathbb{C}) \))**

\[ \mathfrak{g}^\vee \cong \mathbb{C}^3_{e,f,h}, \quad \mathfrak{g} \cong \mathfrak{g}^\vee \text{ via } A \mapsto \text{Tr}(A-) \]

\[ 0 \neq A, B \text{ in same leaf } \iff \det(A) = \det(B) \]

leaves: \( \{0\}, \{ h^2 + 4ef = \text{constant}\} \)
Symmetries

\( \pi \) is invariant under Hamiltonian flows: \( L_{\xi_f} \pi = 0 \), indeed

\[
L_{\xi_f} \pi = [\xi_f, \pi] = [l_{df} \pi, \pi] = -[[f, \pi], \pi] = -[f, [\pi, \pi]] - [\pi, l_{df} \pi] = -L_{\xi_f} \pi = 0
\]

**Definition**

A vector field \( \xi \in T_X \) is a **Poisson vector field** if \( L_{\xi} \pi = 0 \).

**NB:** \( \xi \in T_X \) locally Hamiltonian \( \implies \xi \) Poisson

**Lemma**

If \( \pi \) nondegenerate, then \( \xi \in T_X \) Poisson \( \iff \xi \) is locally Hamiltonian.

**Proof.**

Given \( \xi \in T_X \). Nondegeneracy \( \implies \xi = \pi^\#(\alpha) \) where \( \alpha \in \Omega_X^1 \).

Computation \( \implies L_{\xi} \pi = 0 \) if and only if \( d\alpha = 0 \). Poincaré lemma \( \implies \) locally \( \alpha = df \), so \( \xi = \xi_f \).
**Non-Hamiltonian symmetries**

### Example (locally Hamiltonian $\not\Rightarrow$ globally Hamiltonian)

\[
\pi \text{ nondegenerate, } \alpha \in H^0(\Omega^1_X) \text{ closed but not exact } \implies \xi = \pi^\#(\alpha)
\]

Poisson but not globally Hamiltonian.

### Example (Poisson $\not\Rightarrow$ locally Hamiltonian)

\[
\begin{align*}
X &= \mathbb{C}^3 \quad \text{coords } p, q, z \\
\pi &= \partial_q \wedge \partial_p
\end{align*}
\]

Poisson vector field $\xi = \partial_z$ transverse to leaves

### Example (locally Hamiltonian near a point $\not\Rightarrow$ locally Hamiltonian everywhere)

\[
\begin{align*}
X &= \mathbb{C}^2_{u,v} \\
\pi &= u \partial_u \wedge \partial_v \\
\xi &= \partial_v \\
\partial_v &= \xi_{\log u} \text{ for } u \neq 0 \text{ but not } u = 0.
\end{align*}
\]
Modular vector field

For $X$ smooth, $\dim X = n$, have symmetries generated by volume forms:

$$0 \neq \mu \in \Omega^n_X = K_X$$

\[
\begin{array}{ccc}
\Omega^\bullet_X & \sim & \wedge^{n-\bullet} T_X \\
d & & \text{divergence } \Delta_\mu \\
\text{degree } + 1 & & \text{degree } - 1
\end{array}
\]

**Definition (Brylinski–Zuckerman, Polishchuk, Weinstein 1997)**

If $(X, \pi)$ Poisson manifold, $\mu \in K_X$, the vector field

$$\zeta := \Delta_\mu \pi \in T_X$$

is called the **modular vector field** of $\pi$ with respect to $\mu$.

**Meaning:** $\Delta_\mu \pi = 0 \iff \mu$ invariant under all Hamiltonian flows
Properties of the modular vector field

\[ \zeta := \Delta \mu \pi \]

Local expression:

\[ \pi = \sum \pi^i j \partial_{x_i} \wedge \partial_{x_j} \quad \mu = dx^1 \wedge \cdots \wedge dx^n \quad \zeta = \sum_{ij} \frac{\partial \pi}{\partial x^i} \partial_{x_j} \]

It is a Poisson vector field:

\[ \mathcal{L}_\zeta (\pi) = 0 \]

Change of volume form:

\[ \mu \leadsto \mu' = g \mu \]

\[ \zeta \leadsto \zeta' = \zeta' - \xi \log g \]

So \( \zeta \) is “locally well-defined modulo Hamiltonians”
Modular foliation

Introduce

\[ F_{\pi} \subset F_{\pi}^{\text{mod}} \subset T_X \]

where locally

\[ F_{\pi}^{\text{mod}} = F_{\pi} + O_X \cdot \Delta_{\mu_{\pi}} \]

Check:

\[ [F_{\pi}^{\text{mod}}, F_{\pi}^{\text{mod}}] \subset F_{\pi}^{\text{mod}} \]

Definition

The foliation induced by \( F_{\pi}^{\text{mod}} \) is called the \textbf{modular foliation of} \((X, \pi)\).

Locally: leaves \( L \) of \( F_{\pi}^{\text{mod}} = \) orbits of symplectic leaves under \( \zeta \)

- \( \dim L \) even: symplectic leaf to which \( \zeta \) is tangent
- \( \dim L \) odd: one-parameter family of symplectic leaves related by \( \zeta \)
Examples

Example (Nondegenerate)
\[ \pi \text{ nondegenerate}, \omega = \pi^{-1} \Rightarrow \mathcal{F}_\pi = \mathcal{F}_\pi^{mod} = T_X. \]

NB: \( \mu = \omega^n \in H^0(K_X) \) Hamiltonian-invariant, so \( \zeta = \Delta_{\mu} \pi = 0 \) globally

Example (Constant rank)
\[ \text{rank}(\pi) \text{ constant} \Rightarrow \text{locally } \pi = \sum_i \partial_{q^i} \wedge \partial_{p_i} \Rightarrow \zeta = 0 \text{ in these coords} \Rightarrow \mathcal{F}_\pi^{mod} = \mathcal{F}_\pi \]

Example (\( g = \mathfrak{sl}(2, \mathbb{C}) \))
\[ X = g^\vee = \mathbb{C}^3_{e,f,h} \quad \pi = h\partial_e \wedge \partial_f + 2e\partial_h \wedge \partial_e - 2f\partial_h \wedge \partial_f \]
\[ \mu = de \wedge df \wedge dh \quad \Delta_{\pi} \mu = 0 \]
\[ \mathcal{F}_\pi^{mod} = \mathcal{F}_\pi \quad \text{coadjoint orbits} \]
X smooth connected Poisson surface, $Y = \text{Zeros}(\pi) \subset X$ anticanonical

$$\pi = f(u, v)\partial_u \wedge \partial_v \quad \zeta = f_u\partial_v - f_v\partial_u$$

e.g. if $f = uv$:

In general, modular leaves $L \subset X$:

- $\dim L = 2$: symplectic leaf $L = X \setminus Y$
- $\dim L = 1$: connected components of smooth locus $Y \setminus Y_{\text{sing}}$
- $\dim L = 0$: singular points of $Y$