NORMAL AND NON-NORMAL NUMBERS

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We fix a positive integer $q \geq 2$. Then every real number $x \in [0,1]$ admits a representation of the form

$$x = \sum_{n \geq 1} \frac{a_n}{q^n},$$

where $a_n \in \mathcal{N} := \{0, 1, \ldots, q-1\}$ for $n \geq 1$. For given $x \in [0,1]$, $N \geq 1$, and $\mathbf{d} = d_1 \ldots d_k \in \mathcal{N}^k$ we denote by $\Pi(x, \mathbf{d}, N)$ the frequency of occurrences of the block $\mathbf{d}$ among the first N digits of x, *i.e.*

$$\Pi(x, \mathbf{d}, N) := \frac{1}{N} \left| \{0 \leq n < N \colon a_{n+1} = d_1, \ldots a_{n+k} = d_k\} \right|.$$

From a probabilistic point of view we would expect that in a randomly chosen $x \in [0,1]$ each block $\mathbf{d}$ of k digits occurs with the same frequency q^{-k}. In this respect we call a real $x \in [0,1]$ normal to base q if $\Pi(x, \mathbf{d}, N) = q^{-k}$ for each $k \geq 1$ and each $|\mathbf{d}| = k$. When Borel [1] introduced this concept he could show that almost all (with respect to Lebesgue measure) reals are normal in all bases $q \geq 2$ simultaneously. However, still today all constructions of normal numbers have an artificial touch and we do not know whether given reals such as $\sqrt{2}$, $\log 2$, e or π are normal to a single base.

On the other hand the set of non-normal numbers is large from a topological point of view. We say that a typical element (in the sense of Baire) $x \in [0,1]$ has property P if the set $S := \{x \in [0,1] \colon x \text{ has property } P\}$ is residual – meaning the countable intersection of dense sets. The set of non-normal numbers is residual.

In the present talk we will consider the construction of sets of normal and non-normal numbers with respect to recent results on absolutely normal and extremely non-normal numbers.

References

[1] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques.*, Palermo Rend. **27** (1909), 247–271 (French).

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