

## A Central Limit Theorem for $\mathcal{B}$ -free dynamical systems

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For a set  $\mathcal{B} \subseteq \mathbb{N} \setminus \{1\}$ , let  $\mathcal{B}$ -free integers be the set of positive integers that are not divisible by any element of  $\mathcal{B}$  and let  $X^{\mathcal{B}} \subset \{0, 1\}^{\mathbb{Z}}$  be the closure of the orbit of the indicator of  $\mathcal{B}$ -free integers under the left shift  $T$ . One can equip  $(X^{\mathcal{B}}, T)$  with the  $T$ -invariant measure which produces the correlations of  $\mathcal{B}$ -free integers. For an infinite, coprime  $\mathcal{B}$  with  $\sum_{b \in \mathcal{B}} 1/b < \infty$ , the resulting dynamical system is measurably isomorphic to an ergodic shift on a compact abelian group (el Abdalaoui, Lemańczyk & de la Rue). In particular, for any such set  $\mathcal{B}$ , the corresponding dynamical system is ergodic but not weakly mixing and has zero Kolmogorov–Sinai entropy. In this talk, we present a new Central Limit Theorem for this system under certain additional arithmetic conditions on the underlying set  $\mathcal{B}$ . The proof of the theorem makes use of Bernstein method and is based on a work of Levin and Merzbach.

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## Baez-Duarte's criterion for Riemann's hypothesis

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We propose an overview of results, due to various authors, related to this necessary and sufficient condition, and obtained since its statement in 2003.

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## Multiplicative Functions and Theory of Multiple Recurrence

Vitaly Bergelson (vitaly@math.ohio-state.edu)

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We will report on recent joint work with Joanna Kułaga-Przymus, Mariusz Lemańczyk and Florian Richter.

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## Automatic sequences satisfy Sarnak's conjecture I

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*Joint talk with Clemens Müllner.*

Sequences produced by a deterministic finite automaton with output have a low complexity. Clemens Müllner proved that such sequences satisfy Sarnak's conjecture. The special case when the automaton is synchronizing had been previously proved by him, Jean-Marc Deshouillers and Michael Drmota. In this talk in two parts, we shall first present automatic sequences and the special case of synchronizing automata and then outline a proof of the general result.

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## Positive entropy subshift uncorrelated to the Möbius function

Tomasz Downarowicz (downar@pwr.edu.pl)

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*Joint work with Jacek Serafin.*

Let  $y$  be a bounded sequence of real numbers such that its average along any infinite arithmetic progression equals zero (for example, the Möbius function satisfies this condition). We construct a subshift  $(\Sigma, T)$  on two symbols (i.e.,  $\Sigma$  is a closed shift invariant subset of  $\{-1, 1\}^{\mathbb{N}}$ ,  $T$  denotes the shift map), with entropy arbitrarily close to  $\log 2$ , which does not correlate with  $y$  in the meaning as in Sarnak’s conjecture: for any continuous function  $f: \Sigma \rightarrow \mathbb{R}$  and any  $x \in \Sigma$ , the sequence  $a_n = f(T^n x)$  is uncorrelated to  $y$ . A similar result is announced in Sarnak’s famous “Three lectures” with credits going to Bourgain, however up to date no examples have been published. So we have decided that the problem is in fact open, and this motivated our work.

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## TBA

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## Hausdorff dimension of a class of three-interval exchange maps

Davit Karagulyan (davitk@kth.se)

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I will talk about a result of Bourgain on Moebius disjointness for a class of Three-interval exchange maps. We will describe a forthcoming paper where we estimate the constants in Bourgain’s paper and subsequently estimate the measure of the set of parameters which satisfy the Diophantine condition of Bourgain. As a consequence we show that it has zero Lebesgue measure, but positive Hausdorff dimension. We also give bounds on the Hausdorff dimension.

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## $\mathcal{B}$ -free dynamics – a view through the window

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*Christoph Richard, Stanisław Kasjan, Mariusz Lemańczyk*

All  $\mathcal{B}$ -free dynamical systems can be obtained as weak model sets from a cut-and-project scheme – and can thus be seen through the window defining the weak model set. I will describe a recent fresh view at the cut-and-project schemes (joint work with C. Richard), specialized to the more restricted setting of  $\mathcal{B}$ -free systems, and discuss some aspects of  $\mathcal{B}$ -free systems that become particularly apparent from this point of view (work in progress with S. Kasjan and M. Lemańczyk). Special attention is paid to the unique minimal subsystem (which, in general, need not be trivial) and its maximal equicontinuous factor.

## References

- [1] G. Keller and C. Richard, *Dynamics on the graph of the torus parametrisation*, Published online in Ergodic Theory Dynam. Systems, 2016, <https://doi.org/10.1017/etds.2016.53>

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## Möbius disjointness along ergodic sequences for uniquely ergodic actions

Joanna Kułaga-Przymus (joanna.kulaga@gmail.com)

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*Joint work with Mariusz Lemańczyk.*

Let  $X = \mathbb{T} = [0, 1) \pmod{1}$  and let  $Tx = x + \alpha$  be an irrational rotation. Suppose that  $\varphi: X \rightarrow \mathbb{R}$  is measurable and  $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$  is an ergodic flow acting on a probability standard Borel space  $(Y, \mathcal{C}, \nu)$ . We study so-called Rokhlin extensions of  $T$ , i.e. systems the form  $T_{\varphi, \mathcal{S}}: X \times Y \rightarrow X \times Y$ ,  $T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}y)$ . Under

some additional assumptions on  $\varphi$  (e.g. when  $T_\varphi$ , given by  $T_\varphi(x, t) = (Tx, t + \varphi(x))$ , is ergodic and  $\mathcal{S}$  is weakly mixing), the  $T_{\varphi, \mathcal{S}}$  is relatively weakly mixing over  $T$  [2, 3].

We show that there are irrational  $\alpha \in \mathbb{T}$  and a measurable (even smooth)  $\varphi: X \rightarrow \mathbb{R}$  such that  $T_{\varphi, \mathcal{S}}$  has so-called AOP property (Asymptotic Orthogonality of Powers) [1] for each ergodic flow  $\mathcal{S}$ . In particular, for  $\varphi$  and  $\mathcal{S}$  that are continuous,  $T_{\varphi, \mathcal{S}}$  satisfies Sarnak’s conjecture. This enables us, by taking  $c_n := \varphi(x_0) + \varphi(Tx_0) + \dots + \varphi(T^{n-1}x_0)$ , to find sequences  $(c_n)$  such that  $\sum_{n \leq N} g(S_{c_n}y)\boldsymbol{\mu}(n) = o(N)$  for each uniquely ergodic flow  $\mathcal{S}$  on a compact metric space  $Y$ ,  $g \in C(Y)$  and  $y \in Y$ . In fact, we not only have disjointness with respect to  $\boldsymbol{\mu}$ , but also to any multiplicative function  $\mathbf{u}$  bounded by 1, whenever  $\int g d\nu = 0$ . In fact, since the AOP property holds, a property similar to the one from Sarnak’s conjecture holds on a typical short interval, cf. [1]. More precisely, we have the following: There are Poincaré sequences  $(a_n)_{n \in \mathbb{N}} \subset \mathbb{N}$  such that for each  $\alpha \in [0, 1)$  and each multiplicative function

$$\frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq h < m+H} b(R^{a_h}z)e^{2\pi i a_h \alpha} \mathbf{u}(h) \right| \rightarrow 0, \quad H \rightarrow \infty, \quad H/M \rightarrow 0$$

for each uniquely ergodic homeomorphism  $R$  of a compact metric space  $Z$  ( $b \in C(Z)$ ) of zero average,  $z \in Z$ ).

## References

- [1] H. el Abdalaoui, M. Lemańczyk, T. de la Rue, *Asymptotic orthogonality of powers for quasi-discrete spectrum automorphisms*, International Math. Res. Notices 2016 (online), arXiv: 1507.04132.
- [2] M. Lemańczyk, F. Parreau, *Rokhlin extensions and lifting disjointness*, Ergodic Theory Dynam. Systems **23** (2003), 1525–1550.
- [3] M. Lemańczyk, F. Parreau, *Lifting mixing properties by Rokhlin cocycles*, Ergodic Theory and Dynamical Systems **32** (2012), 763–784.

## Automatic sequences satisfy Sarnak’s conjecture II

**Clemens Müllner** (clemens.muellner@tuwien.ac.at)

Technische Universität Wien, Austria

*Joint talk with Jean-Marc Deshouillers.*

Sequences produced by a deterministic finite automaton with output have a low complexity. Clemens Müllner proved that such sequences satisfy Sarnak’s conjecture. The special case when the automaton is synchronizing had been previously proved by him, Jean-Marc Deshouillers and Michael Drmota. In this talk in two parts, we shall first present automatic sequences and the special case of synchronizing automata and then outline a proof of the general result.

## Möbius function: on the missing log-factor and alia

**Olivier Ramare** (olivier.ramare@univ-amu.fr)

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## Möbius disjointness on short intervals for models of a measure-theoretic dynamical system

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*El Houcein el Abdalaoui, Joanna Kułaga-Przymus and Mariusz Lemańczyk*

Let  $\boldsymbol{\mu} = (\boldsymbol{\mu}(n))_{n \geq 0}$  be the Möbius arithmetic function. Sarnak’s conjecture asserts that any topological dynamical system  $(X, T)$  with zero topological entropy is *Möbius disjoint*, in the sense that for each  $x \in X$  and each  $f \in C(X)$ , we have

$$\frac{1}{N} \sum_{0 \leq n < N} f(T^n x)\boldsymbol{\mu}(n) \xrightarrow{N \rightarrow \infty} 0. \tag{1}$$

In this talk, based on a joint work in progress with El Houcein el Abdalaoui, Joanna Kułaga-Przymus and Mariusz Lemańczyk, I will consider Sarnak’s conjecture from a measure-theoretic point of view, starting from the following general question: when does each uniquely ergodic model of a given measure-theoretic dynamical system  $(Y, \nu, S)$  enjoy the Möbius disjointness property?

We have shown in [1] that this happens if  $(Y, \nu, S)$  has the so-called *AOP property* (Asymptotic Orthogonality of Powers). In a very recent paper [2], Huang, Wang and Zhang have shown that this also happens for a system having discrete spectrum. Putting together some ideas from the two aforementioned works, I will show how the AOP property, or the simple existence of one “good” topological model for  $(Y, \nu, S)$ , implies in fact a stronger property, *Möbius disjointness on short intervals*: for each uniquely ergodic model  $(X, T)$  of  $(Y, \nu, S)$ , for each  $x \in X$  and each  $f \in C(X)$ , for each increasing sequence of integers  $0 = b_0 < b_1 < b_2 < \dots$  with  $b_{k+1} - b_k \rightarrow \infty$ ,

$$\frac{1}{b_K} \sum_{0 \leq k < K} \left| \sum_{b_k \leq n < b_{k+1}} f(T^n x) \mu(n) \right| \xrightarrow{K \rightarrow \infty} 0. \quad (2)$$

Some applications and examples will be given.

## References

- [1] El Houcein el Abdalaoui, Mariusz Lemańczyk and Thierry de la Rue. *Automorphisms with quasi-discrete spectrum, multiplicative functions and average orthogonality along short intervals*, International Mathematics Research Notices, 2016.
- [2] Wen Huang, Zhiren Wang and Guohua Zhang. *Möbius disjointness for topological models of ergodic systems with discrete spectrum*, arXiv:1608.08289v2, 2016.

## Möbius Randomness and Dynamics six years later

**Peter Sarnak** (sarnak@math.ias.edu)

Institute for Advanced Study, USA

There have many developments on the disjointness conjecture of the Möbius (and related) function to topologically deterministic sequences. We review some of these highlighting some related arithmetical questions.

## Effective equidistribution: low-dimensional orbits

**Ilya Vinogradov** (ivinogra@math.princeton.edu)

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I will present a survey of recent results about effective equidistribution of low-dimensional orbits on homogeneous spaces. These include horocycle lifts to bundles with base  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$ , whose fiber is the two-dimensional torus or the Heisenberg manifold. A discussion of number-theoretic applications of these results will follow.

## Möbius disjointness for topological models of ergodic systems with discrete spectrum

**Zhiren Wang** (zhirenw@psu.edu)

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*Joint work with Wen Huang and Guohua Zhang.*

In this talk we will describe a criterion for a point satisfying the required disjointness condition in Sarnak’s Möbius Disjointness Conjecture. As a direct application, we have that the conjecture holds for any topological model of an ergodic system with discrete spectrum.