TILING DYNAMICAL SYSTEM

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Talks

Natalie Priebe Frank
Introduction to hierarchical tiling dynamical systems

Abstract: These lectures introduce the dynamical systems approach to tilings of Euclidean space, especially quasicrystalline tilings that have been constructed using a 'supertile method'. Because tiling dynamics parallels one-dimensional symbolic dynamics, we discuss this case as well, highlighting the differences and similarities in the methods of study and the results that can be obtained.

In the first lecture we motivate the field with the discovery of quasicrystals, which led to D. Schectman's winning the 2011 Nobel Prize in Chemistry. Then we set up the basics of tiling dynamics, describing tiling spaces, a tiling metric, and the shift or translation actions. Shift-invariant and ergodic measures are discussed, along with fundamental topological and dynamical properties.

The second lecture brings in the supertile construction methods, including symbolic substitutions, self-similar tilings, S-adic systems, and fusion rules. Numerous examples are given, most of which are not the "standard" examples, and we identify many commonalities and differences between these interrelated methods of construction. Then we compare and contrast dynamical results for supertile systems, highlighting those key insights that can be adapted to all cases.

In the third lecture we investigate one of the many current tiling research areas: spectral theory. Schectman made his Nobel-prize-winning discovery using diffraction analysis, and studying the mathematical version has been quite fruitful. Spectral theory of tiling dynamical systems is also of broad interest. We describe how these types of spectral analysis are carried out, give examples, and discuss what is known and unknown about the relationship between dynamical and diffraction analysis. Special attention is paid to the "point spectrum", which is related to eigenfunctions and also to the bright spots that appear on diffraction images.

Emmanuel Jeandel
The Undecidability of the Domino Problem

Abstract: One of the most fundamental problem in tiling theory is to decide, given a surface, a set of tiles and a tiling rule, whether there exist a way to tile the surface using the set of tiles and following the rules. As proven by Berger in the 60's, this problem is undecidable in general.

When formulated in terms of tilings of the discrete plane by unit tiles with colored constraints, this is called the Domino Problem and was introduced by Wang in an effort to solve satisfaction problems for $\forall\exists\forall$ formulas by translating the problem into a geometric problem.

In this course, we will give a brief description of the problem and to the meaning of the word "undecidable", and then give two different proofs of the result.

Johannes Kellendonk

Operators, Algebras and their Invariants for Aperiodic Tilings

Abstract: Tilings are a convenient way to model aperiodic structures in space, like quasicrystals or wave guides. The goal of this course is to explain the construction of operators which depend in a local way on tilings, of algebras which contain all these operators, and of topological invariants for them. Through their algebras we thus get topological invariants for tilings. These are K-groups. Furthermore, the K-group elements define topological invariants for self-adjoint operators which have a gap in their spectrum. The interest in such operators comes from physics where they describe particle motion or wave phenomena in the aperiodic structure and, when gapped, topological insulators. If time permits, we explain how the algebraic topology of K-theory gives rise to a bulk-boundary correspondance between the invariants. Here is a rough overview:

In the first lecture we explain the topology of tilings spaces, which is a cross-over between the euclidean topology of the space in which the tiling lies and the locality of interaction principle of physics. We describe the concept of pattern equivariance which is a manifestation of the locality principle. We discuss how to construct pattern equivariant operators on tilings.

In the second lecture we consider the algebras defined by tilings. These arrise through a standard construction from the topological dynamical systems defined by the tiling. The algebras can be thought of as the natural space in which (families of) pattern equivariant operators live. This point of view on the operators leads to results on the spectral continuity of these (families of) operators.

In the third lecture we give a rough overview on the K-theory of the algebras and show how the K-group elements define topological invariants for gapped operators. Using the example of the Toeplitz extension and its associated 6-term exact sequence in K-theory we explain how gap-labels (bulk topological invariants) can be equated with rotation numbers (boundary topological invariants).

This work is based on Jean Bellissard's approach to solid state physics via non-commutative topology. It has been refined over the years and recently raised anewed interest due to the importance of the bulk-boundary correspondance principle for topological insulators. Its relevance to physics lies in the description of topological effects in quantum systems, a research field for which the 2016 Nobel prize in physics was awarded.

Michel Rigo

From combinatorial games to shape-symmetric morphisms

Abstract: The general aim of these lectures is to present some interplay between combinatorial game theory (CGT) and combinatorics on (multidimensional) words.

In the first introductory lecture, we present some basic concepts from combinatorial game theory (positions of a game, Nim-sum, Sprague-Grundy function, Wythoff's game, ...). We also review some concepts from combinatorics on words. We thus introduce the well-known k-automatic sequences and review some of their characterizations (in terms of morphisms, finiteness of their k-kernel,...). These sequences take values in a finite set but the Sprague-Grundy function of a game, such as Nim of Wythoff, is usually unbounded. This provides a motivation to introduce k-regular sequences (in the sense of Allouche and Shallit) whose k-kernel is not finite, but finitely generated.

In the second lecture, games played on several piles of token naturally give rise to a multidimensional setting. Thus, we reconsider k-automatic and k-regular sequences in this extended framework. In particular, determining the structure of the bidimensional array encoding the (loosing) P-positions of the Wythoff's game is a long-standing and challenging problem in CGT. Wythoff's game is linked to non-standard numeration system: P-positions can be determined

by writing those in the Fibonacci system. Next, we present the concept of shape-symmetric morphism: instead of iterating a morphism where images of letters are (hyper-)cubes of a fixed length k, one can generalize the procedure to images of various parallelepipedic shape. The shape-symmetry condition introduced twenty years ago by Maes permits to have well-defined fixed point.

In the last lecture, we move to generalized numeration systems: abstract numeration systems (built on a regular language) and link them to morphic (multidimensional) words. In particular, pictures obtained by shape-symmetric morphisms coincide with automatic sequences associated with an abstract numeration system. We conclude these lectures with some work in progress about games with a finite rule-set. This permits us to discuss a bit Presburger definable sets.

Boris Solomyak
Lecture on Delone sets and Tilings

Abstract: In this lecture we focus on selected topics around the themes: Delone sets as models for quasicrystals, inflation symmetries and expansion constants, substitution Delone sets and tilings, and associated dynamical systems.

Jörg Thuswaldner
*S-**adic sequences***
A bridge between dynamics, arithmetic, and geometry

Abstract: Based on work done by Morse and Hedlund (1940) it was observed by Arnoux and Rauzy (1991) that the classical continued fraction algorithm provides a surprising link between arithmetic and diophantine properties of an irrational number α, the rotation by α on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and combinatorial properties of the well known Sturmian sequences, a class of sequences on two letters with low subword complexity.

It has been conjectured since the early 1990ies that this correspondence carries over to generalized continued fraction algorithms, rotations on higher dimensional tori, and so-called S-adic sequences generated by substitutions. The idea of working towards this generalization is known as *Rauzy's program*. Although, starting with Rauzy (1982) a number of examples for such a generalization was devised, Cassaigne, Frernczi, and Zamboni (2000) came up with a counterexample that showed the limitations of such a generalization.

Nevertheless, recently Berthé, Steiner, and Thuswaldner (2016) made some further progress on Rauzy's program and were able to set up a generalization of the above correspondences. They proved that the above conjecture is true under certain natural conditions. A prominent role in this generalization is played by tilings induced by generalizations of the classical Rauzy fractal introduced by Rauzy (1982).

Another idea which is related to the above results goes back to Artin (1924), who observed that the classical continued fraction algorithm and its natural extension can be viewed as a Poincaré section of the geodesic flow on the space $SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R})$. Arnoux and Fisher (2001) revisited Artin's idea and showed that the above mentioned correspondence between continued fractions, rotations, and Sturmian sequences can be interpreted in a very nice way in terms of an extension of this geodesic flow which they called the scenery flow. Currently, Arnoux *et al.* are setting up elements of a generalization of this connection as well.

It is the aim of my series of lectures to review the above results.

Short presentation

José Manuel Rodriguez Caballero

Balanced parentheses and the E-polynomials of the Hilbert scheme of n points on a torus

Abstract:

L. Göttsche (1999) stablished a relationship between the E-polynomial of a Hilbert scheme of points on a surface and theta functions. Hausel, Letellier and Rodriguez-Villegas (2013) showed that the E-polyomial of the Hilbert scheme of n points on a torus, denoted $C_n(q)$, can be obtained from a product of theta functions which goes back to L. Kronecker (1881). This identity can be used in order to obtain an explicit formula for $C_n(q)$. Kassel and Reutenauer (2015) considered the polynomial $P_n(q) := C_n(q)/(q-1)^2$, whose coefficients are non-negative integers, and they made the following observation: $P_n(q)$ has a coefficient larger than 1 provided that n is a perfect number or an abundant number. We improved this observation in order to obtain a necessary and sufficient condition, which is related to Pythagorean triangles. Surprisingly enough, our proof is based on language theory (balanced parentheses). New connections between the coefficients of the polynomials $P_n(q)$ and the work of famous number theorists like Erdos, Hirschhorn and Tao, can be obtained by means of this language-theoretic approach.

Paulina Cecchi

Congruent monotilable amenable groups and invariant measures

Abstract: This talk we will be about tilings of amenable groups. We will introduce the notion of congruent monotileable amenable group and present a result which states that for any such a group G and any Choquet simplex K, K can be viewed as the set of invariant measures of a minimal G-subshift.

Jungwon Lee

Distribution of modular symbols: a dynamical approach

Abstract:

We give a brief exposition of Baladi and Vallée?s work on dynamics of continued fraction, and how this result can be used to describe the distribution of modular symbols and L-values.

Ivan Mitrofanov

Algorithmical properties of transducer groups and tilings

Abstract:

A set of Wang tiles is called *NW-complete* if for every pair of colours it contains precisely one tile with these North and West colours. We discuss some algorithmical undecidability results about transducer and self-similar groups and translate them to tiling problems about complete tilesets.

Filipp Rukhovich

Outer billiards outside regular polygons: sets of full measure and aperiodic points

Abstract:

For any convex figure, outer billiard map T can be defined as following. Let A be the figure, and x is the point outside it. There are two tangent lines to A containing x; choose one of them, say "right"; let y be point of tangency. Then, $T(x)$ is a point so that y is middle of segment connecting x and $T(x)$.

In case when A is polygon, set of points outside A can be divided into three sets:

(1) set of points with "finite"; points, i.e. points so that T^n is not defined for some integer n;
(2) set of periodic points ($T^n(x) = x$) for some integer n);
(3) set of aperiodic points.

There are two open problems:

- for which n does outer billiard outside regular n-gons have aperiodic points;
- for which n does outer billiard outside regular n-gons have almost all orbits periodic.

It can be easily proved that outer billiards outside triangle, square and regular hexagon don't have aperiodic points. In 1993, S.Tabachnikov proved that outer billiard outside regular pentagon has such points, and set of aperiodic points has fractal structure. On the other hand, almost all points in this case are periodic.

We'll discuss a computer proof of an existence of aperiodic points for outer billards outside regular octagon and dodecagon, and proof of the fact that almost all orbits for outer billiard outside regular octagon are periodic.

Mao Shinoda

uncountably many ergodic maximizing measures for dense continuous functions

Abstract: The purpose of ergodic optimisation is to describe maximising measures which maximise the space average of a function. In the context of thermodynamic formalism they appear as the zero temperature limit of equilibrium measures. Uniqueness of maximising measure for generic continuous function is proved by Jenkinson. On the other hand I will present in this talk the existence of uncountably many ergodic maximising measures for dense continuous functions.

Yotam Smilansky

Kakutani's splitting procedure for substitution partitions

Abstract:

In 1975, S. Kakutani introduced a splitting procedure which generates a sequence of partitions of the unit interval [0,1], and showed that this sequence is uniformly distributed in [0,1]. We present generalizations of this procedure in higher dimensions, which correspond to constructions used when defining substitution and multiscale substitution tilings of Euclidean space. We prove uniform distribution of these sequences of partitions using new path counting results on graphs and establish Kakutani's result as a special case.

Yuki Takahashi

Products of two Cantor sets and application to the Labyrinth model

Abstract:

We consider products of two Cantor sets, and obtain the optimal estimates in terms of their thickness that guarantee that their product is an interval. This problem is motivated by the fact that the spectrum of the Labyrinth model, which is a two dimensional quasicrystal model, is given by a product of two Cantor sets. We also discuss the connection with the question on the structure of intersections of two Cantor sets, which was considered by many authors previously.

Solomom Yaar

Multiscale substitution tilings

Abstract: We present a new scheme to define tilings of Euclidean spaces. These tilings have only finitely many classes of tiles up to a similarity map, but infinitely many isometry-equivalent classes. This is a work in progress, jointly with Yotam Smilansky.

Shuqin Zhang

The space-filling curve of self-similar sets: two examples

Abstract:

Space-filling curves have fascinated mathematicians for over a century since Peano's monumental work in 1890. In a series of four papers, we develop a theory to construct space-filling curves of self-similar sets. We introduce the notion of skeleton for a self-similar set. From a skeleton, we construct several graphs and construct substitution rules from these graphs. Here we use two examples to show our ideas.
