Fano Foliations 0 - Algebraicity of smooth formal schemes and applications to foliations

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May 7, 2020
Plan for the mini-course

- Lecture 0: Algebraicity of smooth formal schemes and applications to foliations
- Lecture 1: Definition, examples and first properties (by C. Araujo)
- Lecture 2: Adjunction formula and applications
- Lecture 3: Classification of Fano foliations of large index (by C. Araujo)
Algebraicity of (smooth) formal schemes - Setup

$X$ (projective) variety over a field $K$

$x \in X(K)$

$\hat{X}$ formal completion of $X$ at $x$

$\hat{V} \subseteq \hat{X}$ a formal subscheme, i.e., an increasing sequence of closed subschemes of $X$ with support $\{x\}$

$\{x\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i \subseteq \cdots \subseteq X$

and $\hat{V} = \varinjlim V_i$ (in the category of ringed spaces over $K$).

The dimension of $\hat{V}$ is the Krull dimension of the $K$-algebra $\varprojlim H^0(X, O_{V_i})$ of regular functions on $\hat{V}$.

Say that $\hat{V}$ is smooth if the $K$-algebra $\varprojlim H^0(X, O_{V_i})$ is isomorphic to $K[[t_1, \ldots, t_N]]$ with $N = \dim \hat{V}$. Then we may assume without loss of generality that $H^0(X, O_{V_i})$ identifies with $K[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{i+1}$.

Example

$K = \mathbb{C}$

$V^{an} \subseteq (X^{an}, x)$ germ of smooth complex analytic variety at $x \leadsto$ smooth formal subscheme $\hat{V}$ of $X$ through $x$ given by $O_{V_i} := O_{V^{an}, x}/M_x^{i+1}$ ($\dim \hat{V} = \dim V^{an}$).
**Lemma**

*TFAE*

1. There exists a closed subvariety $V$ of $X$ over $K$ with $x \in V$ and such that $\hat{V}$ is a branch of $V$ through $x$ (a component of the formal completion of $V$ at $x$).
2. The dimension of the Zariski closure of $\hat{V}$ in $X$ is $\dim \hat{V}$.

**Example (The nodal plane cubic curve)**

The formal completion of the ring $K[x, y]/(y^2 - x^2 - x^3)$ at the origin $(x, y)$ is isomorphic to $K[[u, v]]/(uv)$. 
**Proof.**

Let $Y$ be the Zariski closure of $\hat{V}$ in $X$. Then $Y$ is an integral subscheme of $X$ passing through $x$. Moreover, any branch of $Y$ through $x$ has dimension $\dim Y$.

(1) $\implies$ (2)

Then we must have $\hat{V} \subseteq Y \subseteq V$. Moreover, $\hat{V}$ is a branch of $Y$ at $x$. Hence $\dim Y = \dim \hat{V}$.

(2) $\implies$ (1)

Since $\dim Y = \dim \hat{V}$ by assumption and $\hat{V} \subseteq Y$, $\hat{V}$ is a branch of $Y =: V$ through $x$.

**Definition**

Say that $\hat{V}$ is algebraic if the equivalent conditions below are satisfied.

1. There exists a closed subvariety $V$ of $X$ over $K$ with $x \in V$ and such that $\hat{V}$ is a branch of $V$ through $x$ (a component of the formal completion of $V$ at $x$).

2. The dimension of the Zariski closure of $\hat{V}$ in $X$ is $\dim \hat{V}$.
Example

\( f \in K[[t_1, \ldots, t_N]] \) with \( f(0) = 0 \)

\[ \widehat{V} := \text{Graph}(f) \subset \mathbb{A}_{K}^{N+1} \text{ through } 0 \]

formally defined by the principal ideal generated by \( z - f(t_1, \ldots, t_N) \) in \( K[[t_1, \ldots, t_N, z]] \).

Write \( f = f_{\leq i} \mod (t_1, \ldots, t_N, z)^{i+1} \) with \( f_{\leq i} \in K[t_1, \ldots, t_N] \). Let \( V_i \subset \mathbb{A}_{K}^{N+1} \) be the closed subscheme of \( \text{Spec} K[t_1, \ldots, t_N]/(t_1, \ldots, t_N, z)^{i+1} \subset \mathbb{A}_{K}^{N+1} \) defined by the principal ideal generated by \( f_{\leq i} \)

Then \( \widehat{V} \) is algebraic if and only if \( f \) is algebraic over \( K(t_1, \ldots, t_N) \).

Example

\( K = \mathbb{C} \), \( X \) smooth complex variety, \( \mathcal{G} \subset T_X \) regular foliation, \( L \) the germ of leaf of \( \mathcal{G} \) through a point \( x \in X \)

Then \( \widehat{V} \) is algebraic if and only if \( L \) is algebraic.
X projective variety over a field $K$
$x \in X(K)$
$L$ ample line bundle on $X$
$
\hat{V} \subset \hat{X}$ smooth formal subscheme of the formal completion $\hat{X}$ of $X$ at $x$

**Proposition A (Bost 2004)**

The formal scheme $\hat{V}$ is algebraic if and only if there exists $c > 0$ such that, for any positive integer $j$ and any section $s \in H^0(X, L^\otimes j)$ with $s|_{\hat{V}} \neq 0$, the multiplicity $\text{mult}_x(s|_{\hat{V}})$ of $s|_{\hat{V}}$ at $x$ is $\leq cj$.

$(\text{mult}_x(s|_{\hat{V}}) := \max\{i \mid s|_{V^i} \equiv 0\} \in \mathbb{N} \cup \{+\infty\})$
Proof of sufficiency.

Let $Y$ be the Zariski closure of $\hat{V}$ in $X$. Set $d := \dim \hat{V}$.

Since $\hat{V} \subseteq Y$, we have $d \leq \dim Y$.

Need to show that $\dim Y \leq d$.

Enough to show that $\dim H^0(Y, \mathcal{L}^{\otimes j}|_Y) \leq C \cdot j^d$ for some $C > 0$ and any integer $j$ large enough.

Fix $j_0$ (Serre’s vanishing theorem) such that the restriction map

$$H^0(X, \mathcal{L}^{\otimes j}) \to H^0(Y, \mathcal{L}^{\otimes j}|_Y)$$

is surjective for any $j \geq j_0$.

For $i \geq 0$, set

$$F^i H^0(Y, \mathcal{L}^{\otimes j}|_Y) := \{ s \in H^0(Y, \mathcal{L}^{\otimes j}|_Y) \mid \text{mult}_x(s|_{\hat{V}}) \geq i \}$$

$$= \{ s \in H^0(Y, \mathcal{L}^{\otimes j}|_Y) \mid s|_{V_i} \equiv 0 \}. $$

Decreasing filtration on $H^0(Y, \mathcal{L}^{\otimes j}|_Y)$. 
Notice that the restriction map $H^0(Y, \mathcal{L} \otimes^j) \to H^0(Y, \mathcal{L} \otimes^j_{|\hat{V}})$ is injective $\iff$ mult$_x(s_{|\hat{V}}) < +\infty$ if $s \in H^0(Y, \mathcal{L} \otimes^j) \setminus \{0\}$. In other words,

$$\cap_{i \geq 0} F^i H^0(Y, \mathcal{L} \otimes^j) = \{0\}.$$ 

The exact sequence

$$0 \to \mathcal{I}_{V_{i+1}/Y} \to \mathcal{I}_{V_i/Y} \to \mathcal{N}_{V_i/V_{i+1}} \to 0$$

yields an injective map

$$F^i H^0(Y, \mathcal{L} \otimes^j)/F^{i+1} H^0(Y, \mathcal{L} \otimes^j) \to H^0(V_0, \mathcal{N}_{V_i/V_{i+1}} \otimes \mathcal{L} \otimes^j_{|V_0}).$$

Moreover $\mathcal{N}_{V_i/V_{i+1}} \otimes \mathcal{L} \otimes^j_{|V_0} \cong S^i \mathcal{N}_{V_0/V_1} \otimes \mathcal{L} \otimes^j_{|V_0}$, and hence

$$\dim F^i H^0(Y, \mathcal{L} \otimes^j)/F^{i+1} H^0(Y, \mathcal{L} \otimes^j) \leq \binom{d + i - 1}{i}.$$ 

Now, suppose $j \geq j_0$. By assumption

$$F^i H^0(Y, \mathcal{L} \otimes^j) = \{0\} \text{ if } i > cj.$$
Therefore

\[
\dim H^0(Y, L^j_Y) \leq \sum_{i \geq 0} \dim F^i H^0(Y, L^j_Y) / F^{i+1} H^0(Y, L^j_Y)
\]

\[
\leq \sum_{0 \leq i \leq \lfloor c j \rfloor} \binom{d + i - 1}{i}
\]

\[
\sim \frac{c^d}{d!} j^d \quad \text{as } j \text{ goes to infinity.}
\]

**Proposition B**

Let $X$ and $Y$ be complex projective varieties with $Y \subseteq X$, and let $X_1 \subseteq X$ be a dense Zariski open set such that $Y_1 := X_1 \cap Y \subseteq Y_{\text{reg}}$ and such that $Y \setminus Y_1$ has codimension at least 2. Let $\mathcal{L}$ be an ample line bundle on $X$. Let $V_{1}^{\text{an}} \subset X_{1}^{\text{an}}$ be a germ of smooth locally closed analytic submanifold along $Y_{1}^{\text{an}}$.

Then $V_{1}^{\text{an}}$ is algebraic if and only if there exists $c > 0$ such that, for any positive integer $j$ and any section $s \in H^0(X, \mathcal{L}^\otimes j)$ such that $s|_{V_{1}^{\text{an}}} \neq 0$, the multiplicity $\text{mult}_{Y_1}(s|_{V_{1}^{\text{an}}})$ of $s|_{V_{1}^{\text{an}}}$ along $Y_1$ is $\leq cj$.

**Remark**

A similar statement holds for $\hat{V}$ a smooth formal scheme over any field $K$ with support $Y_1$. 
The proof is similar to that of Proposition A. Let \( Z \subset X \) be the Zariski closure of \( V_1^{an} \) in \( X \). We need to show that \( \dim Z = \dim V_1^{an} =: d \). Set \( Z_1 := Z \cap X_1 \).

Let \( V_i \subset Z_1 \) be the subscheme defined by \( \mathcal{I}_{Y_1^{an}/V_1^{an}}^{i+1} \) for \( i \geq 0 \).

For \( i \geq 0 \), set

\[
F^i H^0(Z, \mathcal{L}_Z^\otimes j) = \{ s \in H^0(Z, \mathcal{L}_Z^\otimes j) \mid s|_{V_i} \equiv 0 \}.
\]

\( \rightsquigarrow \) decreasing filtration on \( H^0(Z, \mathcal{L}_Z^\otimes j) \).

For \( j \gg 1 \) and \( i > c_j \), \( F^i H^0(Z, \mathcal{L}_Z^\otimes j) = \{0\} \).

Thus

\[
\dim H^0(Z, \mathcal{L}_Z^\otimes j) \leq \sum_{0 \leq i \leq [c_j]} \dim F^i H^0(Z, \mathcal{L}_Z^\otimes j)/F^{i+1} H^0(Z, \mathcal{L}_Z^\otimes j).
\]

Suppose for simplicity that \( Y_1 = Y \). Then

\[
\dim F^i H^0(Z, \mathcal{L}_Z^\otimes j)/F^{i+1} H^0(Z, \mathcal{L}_Z^\otimes j) \leq \dim H^0(Y, S^i \mathcal{N}_{Y/V}^* \otimes \mathcal{L}_Y^\otimes j).
\]

Asymptotic Riemann-Roch \( \rightsquigarrow \) there exists \( C > 0 \) such that

\[
\dim H^0(Y, S^i \mathcal{N}_{Y/V}^* \otimes \mathcal{L}_Y^\otimes j) \leq C(i + j)^{\dim Y + \text{rank } \mathcal{N}_{Y/V} - 1}.
\]
**Theorem (Bost, Campana - Păun, – 2018)**

Let $X$ be a normal complex projective variety, let $\mathcal{L}$ be an ample Cartier divisor, and let $\mathcal{G} \subseteq T_X$ be a foliation. Suppose that there exists $c > 0$ such that

$$h^0(X, S[i]\mathcal{G}^* \otimes \mathcal{L}^\otimes j) = 0$$

for any positive integer $j$ and any natural number $i$ satisfying $i > cj$.

Then $\mathcal{G}$ has algebraic leaves.

**Remark**

$X$ smooth and $\mathcal{G}$ regular $\Rightarrow$ the condition above can be rephrased as follows: the tautological class on $\mathbb{P}_X(\mathcal{G}^*)$ is not pseudo-effective.
Application to foliations 1 - Proof

$X_1 \subset X_{\text{reg}}$ open set where $\mathcal{G}|_{X_{\text{reg}}}$ is a subbundle of $T_{X_{\text{reg}}}$,
$Z_1 := X_1 \times X_1$ and $Z := X \times X$,
$L_Z := L \otimes L$,
$X_1 =: Y_1 \subset Z_1$ and $X =: Y \subset Z$ diagonal.

$V_1^{\text{an}} \subset Z_1^{\text{an}}$ germ of the analytic graph of $(X_1, \mathcal{G}|_{X_1})$ along the diagonal $Y_1$
$\leadsto Y_1^{\text{an}} \subset V_1^{\text{an}}$ and $\mathcal{N}_{Y_1^{\text{an}}/V_1^{\text{an}}} \cong \mathcal{G}|_{X_1^{\text{an}}}$,
$\mathcal{G}$ is algebraically integrable if and only if $V_1^{\text{an}}$ is algebraic.

Assumption $(X \setminus X_1$ of codimension $\geq 2) \leadsto$

$$h^0(Y_1^{\text{an}}, S^i \mathcal{N}_{Y_1^{\text{an}}/V_1^{\text{an}}} \otimes L_{Y_1^{\text{an}}}^\otimes j) = 0$$

if $i > cj > 0$
$\leadsto \text{mult}_{Y_1}(s|_{V_1^{\text{an}}}) \leq 2cj$ for any positive integer $j$ and any section $s \in H^0(Z, L_Z^\otimes j)$
such that $s|_{V_1^{\text{an}}}$ is non-zero.

(use

$$0 \rightarrow \mathcal{I}_{Y_1^{\text{an}}/V_1^{\text{an}}}^{i+1} \rightarrow \mathcal{I}_{Y_1^{\text{an}}/V_1^{\text{an}}}^i \rightarrow S^i \mathcal{N}_{Y_1^{\text{an}}/V_1^{\text{an}}} \rightarrow 0$$

$\leadsto$ The theorem follows from Proposition B applied to $(Z, Z_1, Y, Y_1, L_Z)$ and $V_1^{\text{an}}$. 
**Theorem (Campana - Păun 2019)**

Let $X$ be a normal $\mathbb{Q}$-factorial complex projective variety, $\alpha \in N_1(X)_\mathbb{R}$ a movable curve class, and $\mathcal{G} \subset T_X$ a foliation on $X$ of positive rank. Suppose that $\mu^\min_\alpha(\mathcal{G}) > 0$.

Then $\mathcal{G}$ is algebraically integrable and the closure of a general leaf is rationally connected.

$\alpha \in N_1(X)_\mathbb{R}$ is called movable if $D \cdot \alpha \geq 0$ for any effective divisor $D$.

$$\mu_\alpha(\mathcal{D}) = \frac{\det(\mathcal{D}) \cdot \alpha}{\text{rank}(\mathcal{D})}, \quad \mathcal{D} \neq 0 \text{ torsion-free}.$$ 

$$\mu^\min_\alpha(\mathcal{G}) := \inf \{ \mu_\alpha(\mathcal{D}) | \mathcal{D} \neq 0 \text{ is a torsion-free quotient of } \mathcal{G} \}.$$
Application to foliations 2 - Proof

Algebraicity - Suppose for simplicity that $X$ is smooth.

Fact (Campana - Peternell - Toma 2011): $\mu^\min_\alpha(S^iG) = i\mu^\min_\alpha(G)$

$L$ ample line bundle on $X$

$h^0(X, S^iG^* \otimes L^\otimes j) \neq 0 \Rightarrow$ there is a non-zero map $S^iG \to L^\otimes j$

$\Rightarrow i\mu^\min_\alpha(G) = \mu^\min_\alpha(S^iG) \leq \mu_\alpha(L^\otimes j) = j\mu_\alpha(L) \leq j(\mu_\alpha(L) + 1)$

Thus

$h^0(X, S^iG^* \otimes L^\otimes j) = 0$

If $i > \frac{1 + \mu_\alpha(L)}{\mu^\min_\alpha(G)} j$ so that the theorem applies

$(\mu^\min_\alpha(G) > 0$ by assumption and $\mu_\alpha(L) \geq 0$ since $\alpha$ is movable).
Theorem (Bost 2001, Bogomolov - McQuillan 2016)

Let $X$ be a normal complex projective variety, and $\mathcal{G}$ a foliation on $X$. Let $C \subset X$ be a complete curve disjoint from the singular loci of $X$ and $\mathcal{G}$. Suppose that the restriction $\mathcal{G}|_C$ is an ample vector bundle on $C$.

Then the leaf of $\mathcal{G}$ through any point of $C$ is an algebraic variety. The closure of a leaf through a general point of $C$ is rationally connected.

(Algebraicity is a consequence of Proposition B)
Thanks!