Mini-course on Fano Foliations

Carolina Araujo (IMPA)

Lecture 3: Classification of Fano foliations of large index
Mini-course on Fano Foliations

Joint with Stéphane Druel (CNRS/Université Claude Bernard Lyon 1)

- Lecture 0: Algebraicity of smooth formal schemes and applications to foliations
- Lecture 1: Definition, examples and first properties
- Lecture 2: Adjunction formula and applications
- Lecture 3: Classification of Fano foliations of large index
Classification of Fano manifolds

Theorem (Kollár-Miyaoka-Mori 1992)
For fixed $n$, Fano manifolds of dimension $n$ form a bounded family

Classification in dimension $\leq 3$ (Iskovskikh & Mori-Mukai 1977-1981)

Definition
The index of a Fano manifold $X$ is

$$i(F) := \max\{ m \in \mathbb{Z} \mid -K_X = mA, \ A \text{ ample} \}$$

Theorem (Kobayashi-Ochiai 1973)
- $i(X) \leq \dim(X) + 1$
- $i(X) = \dim(X) + 1 \iff X \cong \mathbb{P}^n$
- $i(X) = \dim(X) \iff X \cong Q^n \subset \mathbb{P}^{n+1}$
Classification of Fano manifolds

**Theorem (Fujita 1982)**
Classification when $i(X) = \dim(X) - 1$ (del Pezzo manifolds)

**Theorem (Mukai 1992)**
Classification when $i(X) = \dim(X) - 2$ (Mukai manifolds)

**Theorem (Birkar 2016)**
For singular Fano varieties, boundedness still holds if one suitably bounds the singularities ($\epsilon$-lc)
Fano foliations

Problem

For fixed $r$ and $n$, do Fano foliations of rank $r$ on projective manifolds of dimension $n$ form a bounded family?

Necessary condition (proved in lectures 0 and 1)

$\mathcal{F}$  Fano foliation $\implies \exists$ subfoliation $\mathcal{G} \subset \mathcal{F}$ with algebraic and RC leaves

$\implies X$ is uniruled

Definition

The index of a Fano foliation $\mathcal{F}$ on complex projective manifold $X$ is

$$i(\mathcal{F}) := \max \{ m \in \mathbb{Z} \mid -K_\mathcal{F} \sim_\mathbb{Z} mA, \ A \text{ ample} \}$$
**Kobayashi-Ochiai theorem for foliations**

**Theorem (A.-Druel - Kovács 2008)**

\[ \mathcal{F} \subseteq T_X \] Fano foliation of rank \( r \) on a complex projective manifold \( X \)

- \( i(\mathcal{F}) \leq r \)
- \( i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n \)

**Theorem (Wahl 1983)**

\( X \) complex projective manifold

If \( T_X \) contains an ample line bundle, then \( X \cong \mathbb{P}^n \)

\( \implies \) in the theorem we may assume that \( r \geq 2 \)
KOBEYASHI-OCHIAI THEOREM FOR FOLIATIONS

Theorem (A.- Druel - Kovács 2008)
\[ \mathcal{F} \subsetneq \mathcal{T}_X \] Fano foliation of rank \( r \) on a complex projective manifold \( X \)

- \( i(\mathcal{F}) \leq r \)
- \( i(\mathcal{F}) = r \) \( \implies \) \( X \cong \mathbb{P}^n \)

Proof.
Let \( \mathcal{F} \subsetneq \mathcal{T}_X \) be Fano foliation of rank \( r \geq 2 \) and index \( i(\mathcal{F}) \geq r \)

- Step 1. Show that \( i(\mathcal{F}) = r \)
- Step 2. Show that the leaves of \( \mathcal{F} \) are algebraic
- Step 3. Show that the general log leaf \( (F, \Delta) \cong (\mathbb{P}^r, H) \) (log canonical)
- Step 4. Using the common point, show that \( X \cong \mathbb{P}^n \)
**Tool: Rational Curves on Uniruled Varieties**

$X$ complex projective manifold of dimension $n$

$\mathcal{W}$ dominating family of rational curves of minimal degree on $X$ ($\mathcal{W} \subset \text{Chow}(X)$)

$x \in X$ general $\rightsquigarrow \mathcal{W}_x = \{ [\ell] \in \mathcal{W} \mid x \in \ell \}$ proper ($d = \dim(\mathcal{W}_x)$)

**Properties**

- $\forall$ closed subset $Z \subset X$ with $\text{codim}_X(Z) \geq 2$
  $\exists \ \ell \in \mathcal{W}$ such that $\ell \cap Z = \emptyset$

- For general $[\ell] \in \mathcal{W}$, $T_X|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$
  
  $= \mathcal{T}_{\mathbb{P}^1}$
**Tool: Rational Curves on Uniruled Varieties**

$X$ complex projective manifold of dimension $n$

$W$ dominating family of rational curves of minimal degree on $X$

$(W \subset Chow(X))$

$x \in X$ general $\leadsto W_x = \{[\ell] \in W \mid x \in \ell\}$ proper $\quad (d = \dim(W_x))$

**Properties**

- $\forall$ closed subset $Z \subset X$ with $\text{codim}_X(Z) \geq 2$
  $\exists \ell \in W$ such that $\ell \cap Z = \emptyset$

- For general $[\ell] \in W$, $T_{X|\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{d} \oplus \mathcal{O}_{\mathbb{P}^1}^{(n-d-1)}$

**Theorem (Cho-Miyaoka-Shepherd-Barron, Kebekus 2002)**

$d = n - 1 \iff X \cong \mathbb{P}^n \iff \exists x_0 \in X$ such that curves from $W_{x_0}$ dominate $X$
**Rationally connected quotients**

$X$ complex projective manifold

$W$ dominating family of rational curves on $X$

Equivalence relation on $X$:

$x \sim y \iff x$ and $y$ can be connected by a chain of cycles in $\overline{W}$

∃ dense open subset $X^\circ \subset X$ and proper morphism

$$\pi : X^\circ \to Y^\circ$$

whose fibers are equivalence classes

For general $[\ell] \in W$:

$$T_{X|\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d \oplus \mathcal{O}_{\mathbb{P}^1}(n-d-1) \oplus (T_{X^\circ/Y^\circ}|_{\ell})$$
**Rationally connected quotients**

**Remark**

\( X \) complex projective manifold

\( W \) proper (unsplit) family of rational curves on \( X \)

(e.g., for some ample divisor \( A \) on \( X \), \( A \cdot \ell = 1 \), \([\ell]\) \( \in \) \( W \))

\[ x \sim y \iff x \text{ and } y \text{ can be connected by a chain of cycles in } W \]

\( \exists \) dense open subset \( X^\circ \subset X \) with \( \text{codim}_X(X \setminus X^\circ) \geq 2 \) and equidimensional proper morphism onto normal variety

\[ \pi : X^\circ \to Y^\circ \]

whose fibers are equivalence classes, reduced and irreducible
Kobayashi-Ochiai theorem for foliations

Theorem (A.- Druel - Kovács 2008)
$\mathcal{F} \subsetneq T_X$ Fano foliation of rank $r$ on a complex projective manifold $X$

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Proof.
Let $\mathcal{F} \subsetneq T_X$ be Fano foliation of rank $r \geq 2$ and index $i(\mathcal{F}) \geq r$

- Step 1. Show that $i(\mathcal{F}) = r$
- Step 2. Show that the leaves of $\mathcal{F}$ are algebraic
- Step 3. Show that the general log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ (log canonical)
- Step 4. Using the common point, show that $X \cong \mathbb{P}^n$
**Step 1. Show that** \( i(\mathcal{F}) = r \)

**Assumption:** \(-K_{\mathcal{F}} = i(\mathcal{F})A\), \(A\) ample and \(i(\mathcal{F}) > r\)

\(\mathcal{W}\) dominating family of rational curves of minimal degree on \(X\) with associated rationally connected quotient \(\pi : X^\circ \to Y^\circ\)

\([\ell] \in \mathcal{W}\) general \(\implies\) \(\ell \cap \text{Sing}(\mathcal{F}) = \emptyset\) and

\[\mathcal{F}_{|_\ell} \subset T_{X_{|_\ell}} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)}\]

\[\implies\] \[\mathcal{F}_{|_\ell} \cong \underbrace{\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}}_{\text{and } A \cdot \ell = 1 \ (\mathcal{W} \ \text{unsplit})} \quad \text{and}\]

\[\implies\] \[T_{X^\circ/Y^\circ} \subset \mathcal{F}_{|X^\circ}\]

\[\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \subset (T_{X^\circ/Y^\circ})_{|_\ell} \subset \mathcal{F}_{|_\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \subset \]

\[\subset \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)} \cong T_{X_{|_\ell}}\]
**Step 1. Show that** $i(\mathcal{F}) = r$

**Assumption:** $-K_{\mathcal{F}} = i(\mathcal{F})A$, $A$ ample and $i(\mathcal{F}) > r$

$\mathcal{W}$ dominating family of rational curves of minimal degree on $X$ with associated rationally connected quotient $\pi : X^o \to Y^o$

$[\ell] \in \mathcal{W}$ general $\implies \ell \cap \text{Sing}(\mathcal{F}) = \emptyset$ and

$\mathcal{F}|_{\ell} \subset T_{X|\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)}$

$\implies \mathcal{F}|_{\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ and $A \cdot \ell = 1$ (\mathcal{W} unsplit)

$\implies T_{X^o/Y^o} \subset \mathcal{F}|_{X^o}$

$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \cong (T_{X^o/Y^o})|_{\ell} = \mathcal{F}|_{\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \subset$

$\subset \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)} \cong T_{X|\ell}$
Step 1. Show that $i(\mathcal{F}) = r$

Assumption: $-K_{\mathcal{F}} = i(\mathcal{F})A$, $A$ ample and $i(\mathcal{F}) > r$

$\mathcal{W}$ dominating family of rational curves of minimal degree on $X$ with associated rationally connected quotient $\pi: X^{\circ} \to Y^{\circ}$

Conclusion: $\mathcal{F}$ is induced by $\pi: X^{\circ} \to Y^{\circ}$

General log leaf $(F, \Delta) = (X_y, 0)$

Corollary (proved in lecture 2)

If $\mathcal{F}$ is an algebraically integrable Fano foliation on a complex projective manifold, then $\Delta \neq 0$.

Contradiction!
Kobayashi-Ochiai theorem for foliations

Theorem (A.- Druel - Kovács 2008)

\[ \mathcal{F} \subsetneq T_X \] Fano foliation of rank \( r \) on a complex projective manifold \( X \)

- \( i(\mathcal{F}) \leq r \)
- \( i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n \)

Proof.

Let \( \mathcal{F} \subsetneq T_X \) be Fano foliation of rank \( r \geq 2 \) and index \( i(\mathcal{F}) \geq r \)

- Step 1. Show that \( i(\mathcal{F}) = r \)
- Step 2. Show that the leaves of \( \mathcal{F} \) are algebraic
- Step 3. Show that the general log leaf \((F, \Delta) \cong (\mathbb{P}^r, H)\) (log canonical)
- Step 4. Using the common point, show that \( X \cong \mathbb{P}^n \)
**Step 2. Show that leaves are algebraic**

**Assumption:** \(-K_F = rA, \ A \text{ ample}\)

\(\mathcal{W}\) dominating family of rational curves of minimal degree on \(X\)

\(\mathcal{W} \xymatrix{ \ar[r]^-{\alpha} & N_1(X)}\) movable curve class \(\xymatrix{ \ar[r]^-{\mu_\alpha} & \det(\bullet) \cdot \alpha\over \text{rank}(\bullet)}\)

The **Harder-Narasimhan filtration** of \(\mathcal{F}\):

\(0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{F}\)

\(\mu_\alpha(\mathcal{F}_1) > \mu_\alpha(\mathcal{F}_2) > \cdots > \mu_\alpha(\mathcal{F}_k) \geq 1\)

**Theorem (proved in lectures 0 and 1)**

\(\mathcal{F}_1\) has algebraic (and RC) leaves

**Case 1.** \(\mathcal{F} = \mathcal{F}_1\) is \(\mu_\alpha\)-semistable \(\implies\) \(\mathcal{F}\) has algebraic leaves

**Case 2.** \(\mathcal{F}_1 \neq \mathcal{F}\) \(\implies\) \(\mu_\alpha(\mathcal{F}_1) > 1\)
Step 2. Show that leaves are algebraic

Case 2. $\mathcal{F}_1 \subsetneq \mathcal{F}$ with $\mu_\alpha(\mathcal{F}_1) = \frac{\det(\mathcal{F}_1) \cdot \alpha}{\text{rank}(\mathcal{F}_1)} > 1 \implies$ (as in step 1)

- $\mathcal{W}$ unsplit
- $\mathcal{F}_1$ has rank $r - 1$
- $\mathcal{F}_1$ is induced by the rationally connected quotient associated to $\mathcal{W}$

$$\pi : X^\circ \to Y^\circ$$

(\text{codim}_X(X \setminus X^\circ) \geq 2 \text{ and } \pi \text{ equidimensional and proper with reduced and irreducible fibers onto normal variety})

$$\implies \mathcal{F} = \pi^* \mathcal{G} \text{ for } \mathcal{G} \subset T_{Y^\circ} \text{ foliation of rank } 1$$

$$K_{\mathcal{F}} = K_{X^\circ/Y^\circ} + \pi^* K_{\mathcal{G}}$$
Step 2. Show that leaves are algebraic

$X^\circ \subset X$ open subset with $\text{codim}_X(X \setminus X^\circ) \geq 2$

$\pi : X^\circ \to Y^\circ$ equidimensional and proper with reduced fibers

$G \subset T_{Y^\circ}$ foliation of rank 1

$\mathcal{F} = \pi^* G$ ~ $-K_\mathcal{F} = -K_{X^\circ/Y^\circ} - \pi^* K_G$

$\tilde{C} \subset X$ general complete intersection curve $\implies \tilde{C} \subset X^\circ$

$C = \pi(\tilde{C}) \subset Y^\circ$ (we may assume it is smooth) and $X_C = \pi^{-1}(C)$

$\pi_C : X_C \to C$ equidimensional and proper with reduced fibers

$(-K_\mathcal{F}|_{X_C} = \underbrace{-K_{X_C/C}}_{\text{ample}} - \pi^*(K_G|_C)$

$\implies -K_G \cdot C > 0$

$\implies$ leaves of $\mathcal{G}$ are algebraic
**Kobayashi-Ochiai theorem for foliations**

**Theorem (A.- Druel - Kovács 2008)**

\[ \mathcal{F} \subsetneq T_X \text{ Fano foliation of rank } r \text{ on a complex projective manifold } X \]

- \( i(\mathcal{F}) \leq r \)
- \( i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n \)

**Proof.**

Let \( \mathcal{F} \subsetneq T_X \) be Fano foliation of rank \( r \geq 2 \) and index \( i(\mathcal{F}) \geq r \)

- **Step 1.** Show that \( i(\mathcal{F}) = r \)
- **Step 2.** Show that the leaves of \( \mathcal{F} \) are algebraic
- **Step 3.** Show that the general log leaf \( (F, \Delta) \cong (\mathbb{P}^r, H) \) (log canonical)
- **Step 4.** Using the common point, show that \( X \cong \mathbb{P}^n \)
**Step 3. Show that** \((F, \Delta) \cong (\mathbb{P}^r, H)\)

**Assumption:** \(-K_F = rA\), \(A\) ample + leaves are algebraic

\(\sim (F, \Delta)\) general log leaf \((\Delta \neq 0)\)

**Adjunction theory:** To describe a polarized variety \((Y, L)\) by studying

\[K_Y + mL, \quad m \geq 1 \quad (\text{adjunction divisors})\]

**Example (Fujita 1988)**

\[K_Y + \dim(Y)L\] not pseudo-effective \(\implies (Y, L) \cong (\mathbb{P}^n, H)\)

In our case: \((Y, L) = (F, A_F)\)

\[
\begin{align*}
K_F + \Delta &\sim (K_F)_{|F} \sim -rA_F \\
\implies K_F + rA_F &\sim -\Delta \text{ is not pseudo-effective} \\
\implies (F, A_F) &\cong (\mathbb{P}^r, H) \quad \text{and} \quad \Delta \sim H
\end{align*}
\]
Kobayashi-Ochiai theorem for foliations

Theorem (A.-Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank $r$ on a complex projective manifold $X$

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Proof.

Let $\mathcal{F} \subsetneq T_X$ be Fano foliation of rank $r \geq 2$ and index $i(\mathcal{F}) \geq r$

- Step 1. Show that $i(\mathcal{F}) = r$
- Step 2. Show that the leaves of $\mathcal{F}$ are algebraic
- Step 3. Show that the general log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ (log canonical)
- Step 4. Using the common point, show that $X \cong \mathbb{P}^n$
Step 4. Show that $X \cong \mathbb{P}^n$

**Assumption:** $-K_F = rA$, $A$ ample + leaves are algebraic

General log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ and $A_F \sim H$

$\ell \subset F \cong \mathbb{P}^r \leadsto W$ dominating (unsplit) family of rational curves on $X$

**Corollary (proved in lecture 2)**

$\mathcal{F}$ algebraically integrable Fano foliation on a complex projective manifold

If the general log leaf $(F, \Delta)$ is log canonical, then there is a common point in the closure of a general leaf.

$x_0 \in X$ common point in the closure of a general leaf $F \cong \mathbb{P}^r$

Curves from $W_{x_0}$ dominate $X \quad \implies \quad X \cong \mathbb{P}^n \quad \square$
**Theorem (A.-Druel - Kovács 2008)**

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank $r$ on a complex projective manifold $X$

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

**Definition**

A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank $r$ on a complex projective manifold $X$ is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$. 
**del Pezzo foliations**

**Definition**

A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank $r$ on a complex projective manifold $X$ is a del Pezzo foliation if $i(\mathcal{F}) = r - 1$.

**Theorem (A.-Druel 2013)**

If $\mathcal{F}$ is a del Pezzo foliation on a complex projective manifold $X$, then

- either $X \cong \mathbb{P}^n$ and $\exists \varphi : \mathbb{P}^n \to \mathbb{P}^{n-r+1}$ such that $\mathcal{F} = \varphi^* \mathcal{C}$ for $\mathcal{C} \cong \mathcal{O} \subset T_{\mathbb{P}^{n-r+1}}$, or

- $\mathcal{F}$ is algebraically integrable

**Problem**

Classification of del Pezzo foliations
Classification of del Pezzo foliations

Theorem (A.-Druel 2016, A. 2018)

Classification of log leaves \((F, \Delta)\) of del Pezzo foliations on complex projective manifolds:

1. \((F, \Delta) \cong (\mathbb{P}^r, Q^{r-1})\)
2. \((F, \Delta) \cong (Q^r, H)\)
3. \((F, \Delta) \cong (\mathbb{P}^2, \ell)\)
4. \(F \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) + \text{classification of } \mathcal{E} \text{ and description of } \Delta \ (r \leq 3)\)
5. \((F, \Delta) \) is a cone over \((C_d, p_1 + p_2), \) where \(C_d\) is rational normal curve of degree \(d\) in \(\mathbb{P}^d\)
6. \((F, \Delta) \) is a cone over (4)
**Classification of del Pezzo foliations**

**Theorem (A.-Druel 2013, 2016, Figueredo 2019)**

\( \mathcal{F} \) del Pezzo foliation of rank \( r \geq 3 \) on complex projective manifold \( X \not\cong \mathbb{P}^n \)

Suppose that the general log leaf \((F, \Delta)\) is log canonical. Then

- either \( X \cong \mathbb{Q}^n \) and \( \mathcal{F} \) is induced by a linear projection \( \mathbb{P}^{n+1} \to \mathbb{P}^{n-r} \)
- or \( r = 3 \) and \( X \cong \mathbb{P}_{\mathbb{P}^k}(\mathcal{E}) \) ( + classification of \( \mathcal{E} \) and \( \mathcal{F} \) )

**Problem**

*Classification of del Pezzo foliations of rank \( r = 2 \)
Thank you!