1. Fano foliations with large index

1.1. Del Pezzo foliations. The papers [AD13], [AD16] and [Fig19] give a classification of algebraically integrable del Pezzo foliations of rank $\geq 3$ with log canonical singularities (in the sense of [AD13, Definition 3.7]). The following is still missing.

**Question 1.1.** Classify algebraically integrable del Pezzo foliations of rank 2 with log canonical singularities.

The following is the classification of all possible general log leaves of del Pezzo foliations on smooth projective varieties.

**Proposition 1.2 ([Ara16]).** Let $\mathcal{F}$ be an algebraically integrable del Pezzo foliation of rank $r \geq 2$ on a smooth projective variety $X$, with general log leaf $(F, \Delta)$. Let $L$ be an ample divisor on $X$ such that $-K_{\mathcal{F}} \sim (r-1)L$. Then $(F, \Delta, L|_F)$ satisfies one of the following conditions.

1. $(F, \mathcal{O}_F(\Delta), \mathcal{O}_F(L|_F)) \cong (\mathbb{P}^{r}, \mathcal{O}_{\mathbb{P}^{r}}(2), \mathcal{O}_{\mathbb{P}^{r}}(1))$.
2. $(F, \Delta)$ is a cone over $(Q^m, H)$, where $Q^m$ is a smooth quadric hypersurface in $\mathbb{P}^{m+1}$ for some $2 \leq m \leq r$, $H \in \mathcal{O}_{Q^m}(1)$, and $L|_F$ is a hyperplane under this embedding.
3. $(F, \mathcal{O}_F(\Delta), \mathcal{O}_F(L|_F)) \cong (\mathbb{P}^{2}, \mathcal{O}_{\mathbb{P}^{2}}(1), \mathcal{O}_{\mathbb{P}^{2}}(2))$.
4. $(F, \mathcal{O}_F(L|_F)) \cong (\mathbb{P}^{1}(\mathcal{E}), \mathcal{O}_{\mathbb{P}^{1}(\mathcal{E})}(1))$, and one of the following holds:
   a. $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(d)$ for some $d \geq 2$, and $\Delta \sim_{\mathbb{P}} \sigma + f$, where $\sigma$ is the minimal section and $f$ a fiber of $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^{1}$.
   b. $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{1}}(d)$ for some $d \geq 2$, and $\Delta$ is a minimal section.
   c. $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(d)$ for some $d \geq 1$, and $\Delta = \mathbb{P}^{1}(\mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1))$.
5. $(F, \Delta)$ is a cone over $(C_d, B)$, where $C_d$ is rational normal curve of degree $d$ in $\mathbb{P}^d$ for some $d \geq 2$, $B \in \mathcal{O}_{\mathbb{P}^{1}}(2)$, and $L|_F$ is a hyperplane under this embedding.
6. $(F, \Delta)$ is a cone over the pair (4a) above, and $L|_F$ is a hyperplane section of the cone.

**Question 1.3.** Which possible general log leaves of del Pezzo foliations do actually occur for del Pezzo foliations on smooth projective varieties?

See [Ara16, Remark 2.10] for some examples of non-log canonical general log leaves of del Pezzo foliations.

The structure of del Pezzo foliations on $\mathbb{P}^m$-bundle $\pi : X \to \mathbb{P}^d$ is given in [AD13]. Recall from [AD13, Theorem 1.1] that $\mathcal{F}$ is then algebraically integrable with rationally connected general leaves.

If $m = 1$, then $X \simeq \mathbb{P}^1 \times \mathbb{P}^d$, and $\mathcal{F}$ is the pullback via $\pi$ of a foliation $\mathcal{O}(1)^{\oplus i} \subset T_{\mathbb{P}^d}$ for some $i \in \{1, 2\}$.

**Theorem 1.4 ([AD13, Theorem 1.4]).** Let $\mathcal{F} \subset T_X$ be a del Pezzo foliation on a $\mathbb{P}^m$-bundle $\pi : X \to \mathbb{P}^d$, with $m \geq 2$. Suppose that $\mathcal{F} \not\subset T_X$. Then there is an exact sequence of vector bundles $0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{L} \to 0$ on $\mathbb{P}^d$ such that $X \simeq \mathbb{P}^d(\mathcal{E})$, and $\mathcal{F}$ is the pullback via the relative linear projection $X \dashrightarrow Z = \mathbb{P}^d(\mathcal{K})$ of a foliation $q^* \det(\mathcal{L}) \subset T_Z$. Here $q : Z \to \mathbb{P}^d$ denotes the natural projection. Moreover, one of the following holds.
Singularities of linear systems and boundedness of Fano varieties


1.2. Mukai foliations. Let $\mathcal{F}$ be a Mukai foliation of rank $r$ on an $n$-dimensional complex projective manifold $X \not\cong \mathbb{P}^n$ (i.e., the index of $\mathcal{F}$ satisfies $\text{ind}(\mathcal{F}) = r - 2 \geq 1$). We know from [AD19, Corollary 1.6] that the algebraic rank of $\mathcal{F}$ satisfies $r^a_{\mathcal{F}} \geq \text{ind}(\mathcal{F}) + 1$. In [AD17], we have classified codimension 1 Mukai foliations on projective manifolds. It follows from this classification that, in the codimension 1 case, if $r^a_{\mathcal{F}} = \text{ind}(\mathcal{F}) + 1$, then

1. either $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$, and $\mathcal{F}$ is the pullback under the restriction to $X$ of a linear projection $\mathbb{P}^{n+1} \to \mathbb{P}^2$ of a foliation on $\mathbb{P}^2$ induced by a global vector field;

2. or $X$ is a projective space bundle over a curve or a surface, and $\mathcal{F}$ is the pullback of a codimension 1 foliation on a surface or threefold.

Question 1.5. Let $\mathcal{F}$ be a above. Describe the general log leaf of $\mathcal{F}$. Then deduce those foliations with log canonical singularities in the sense of [AD13, Definition 3.7].

Question 1.6. Describe foliations as above with log canonical singularities in the sense of McQuillan.

2. Boundedness for Fano foliations

For Fano varieties, we have the following boundedness results.

Theorem 2.1 ([KMM92]). For fixed $n$, Fano manifolds of dimension $n$ form a bounded family.

If we allow Fano varieties with arbitrary singularities, then boundedness fails, already in dimension 2. Consider for instance cones over rational normal curves of degree $n > 0$. They all have klt singularities, and clearly do not form a bounded family. On the other hand, if one suitably bounds the singularities, then boundedness still holds for singular Fano varieties of fixed dimension. More precisely:

Theorem 2.2 ([Bir16]). For fixed positive integer $n$ and positive real number $\varepsilon$, Fano varieties of dimension $n$ with $\varepsilon$-log canonical singularities form a bounded family.

Question 2.3. For fixed positive integers $r$ and $n$, do Fano foliations $\mathcal{F}$ of rank $r$ on projective manifolds of dimension $n$ form a bounded family, possibly imposing restrictions on the singularities of $\mathcal{F}$?

3. Geometry of webs on projective manifolds

Study global properties of webs (see [PP15] for this notion) following ideas from the minimal model program.

Question 3.1. Define the canonical class of a web.

Question 3.2. Is there a Kobayashi-Ochiai theorem for Fano webs?

Question 3.3. Define notions of singularities for webs.

Question 3.4. Study the geometry of Fano webs with large index (and mild singularities).

References


