Complete holomorphic vector fields and their singular points

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Research School “Geometry and Dynamics of Foliations”
CIRM, 2020
Problem

- Understand holomorphic flows on complex manifolds
- Understand holomorphic actions of complex Lie groups on complex manifolds/analytic spaces
Flows, actions of $\mathbb{C}$, complete vector fields

Complex flow, holomorphic action of $\mathbb{C}$

$M$ complex manifold, $\Phi : \mathbb{C} \times M \to M$

- $\Phi(0, p) = p$
- $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$

Action $\longrightarrow$ complete vector field: for fixed $p \in M$

$$X(p) = \left. \frac{d}{dt} \Phi(t, p) \right|_{t=0}$$
From vector fields to flows

\( M \) manifold, \( X \) vector field on \( M \)
Does there exist a flow inducing \( X \)?

- yes, if \( M \) is compact
- in general, no

Rebelo: there are local obstructions!
For instance:

- Understand (semi)complete polynomial vector fields on affine surfaces (Brunella, G.-Rebelo)
- Complete the classification of compact complex surfaces admitting vector fields (Dloussky-Oeljeklaus-Toma)
- Understand compact complex threefolds with a holomorphic action of $\text{SL}(2, \mathbb{C})$ (G.)
- Understand complex surfaces with a holomorphic action of $\mathbb{C}^2$ (Rebelo-Reis-Ferreira)
A result:

**Theorem (Rebelo, 2007; G., 2014)**

Let $V$ be a two-dimensional normal irreducible Stein space, $X$ a complete holomorphic vector field on $V$, $p \in V$ isolated zero of $X$. Either:

- $p$ is a regular point of $V$, where $X$ is non-degenerate;
- $p$ is a weighted homogeneous singularity ($X$ generates the weighted homotheties); or
- $p$ is a cyclic quotient (Hirzebruch-Jung) singularity ($X$ is the quotient of a non-degenerate vector field).
Locally...

$M$ manifold, $X$ vector field on $M$.

In a chart:

$$X = \sum_{i=1}^{n} f_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i};$$

its integration is equivalent to the system

$$z_i'(t) = f_i(z_1(t)), \ldots, z_n(t)), i = 1, \ldots, n$$

Cauchy

For every $p \in M$

- there exists $U \subset \mathbb{C}$ and $\phi : U \to M$, $\phi(0) = p$ solution to the equation;
- the germ of $U$ at 0 is unique.
Vector fields on curves

$L$ complex curve

Three equivalent data:

- $X$ holomorphic vector field without zeros
- $\omega$ holomorphic form without zeros (time form), $\omega(X) \equiv 1$
- translation structure on $L$:
  - if $\phi : U \to L$ is a solution of $X$, $\phi^{-1}$ is a chart of the translation structure (if $\phi(t)$ is a solution, $\phi(t + c)$ is a solution as well);
  - $D(z) = \int_{z_0}^{z} \omega$ is a chart of the translation structure.
Semicomplete/univalent vector fields

Let $L$ be a curve, $X$ a nowhere vanishing holomorphic vector field on $L$. The following are equivalent:

1. For every solution $\phi : U \to L$ ($U \subset \mathbb{C}$, $0 \in U$) of $X$ and every pair of paths $\gamma_i : [0, 1] \to \mathbb{C}$ ($\gamma_i(0) = 0$, $\gamma_1(1) = \gamma_2(1)$), such that the germ of $\phi$ at 0 admits analytic continuations along $\gamma_1$ and $\gamma_2$, both analytic continuations define the same germ at $\gamma_i(1)$.

2. There exists $\Omega \subset \mathbb{C}$ and $\phi : \Omega \to L$ a solution to $X$ such that $\hat{\phi} : \Omega \to \mathbb{C} \times L$, $\hat{\phi}(t) = (t, \phi(t))$ is proper.

3. There exists $\Omega \subset \mathbb{C}$ and a covering map $\phi : \Omega \to L$ that is a map between curves with translation structures.

4. For every path $\gamma : [0, 1] \to L$ such that $\gamma(0) \neq \gamma(1)$, $\int_{\gamma} dT \neq 0$.

If $X$ satisfies all (any) of these properties, $X$ is univalent or semicomplete.
Semicomplete vector fields (Palais, Rebelo, . . .)

- $M$ manifold
- $X$ holomorphic vector field on $M$
- $\mathcal{F}$ the induced foliation (w/singularities)

**Semicompleteness:**

$X$ is semicomplete on $M$ if for every leaf $L$ of $\mathcal{F}$ in $M \setminus \text{sing}(X)$, $X|_L$ is semicomplete.
The semiglobal flow of a semicomplete vector field

$M$ manifold, $X$ vector field on $M$.

If $X$ is semicomplete...

...there exists $\Omega \subset \mathbb{C} \times M$, $\{0\} \times M \subset \Omega$, $\Phi : \Omega \to M$ such that

1. $\Phi(0, p) = p$;
2. if $(s, p), (t, \Phi(s, p))$ and $(t + s, p)$ are in $\Omega$,
   $$\Phi(t + s, p) = \Phi(t, \Phi(s, p));$$
3. for $\Omega_p = \{t \mid (t, p) \in \Omega\}$, $\Omega_p \xrightarrow{f} \mathbb{C} \times M$, $(t, p) \mapsto (t, \Phi(t))$ is proper;
4. for every $p \in M$,
   $$\left. \frac{d}{dt} \Phi(t, p) \right|_{t=0} = X(p).$$
The semiglobal flow of a semicomplete vector field

Reciprocally...
If there exists $(\Omega, \Phi)$ as before, $X$ is semicomplete.
Semicompleteness: some properties

- Complete vector fields are semicomplete.
- The restriction of a semicomplete vector field to an open subset remains semicomplete.
- Semicomplete vector fields may be germified! (Rebelo).
- On a given manifold, the space of semicomplete vector fields is closed (Poincaré, . . . , Ghys-Rebelo).
Palais’s definition of *univalence* (in the context of Lie algebras of vector fields on manifolds) and Rebelo’s notion of *semicompleteness* appear in:

- **Richard S. Palais.**
A global formulation of the Lie theory of transformation groups.  

- **Julio C. Rebelo.**
Singularités des flots holomorphes.  
The introductions to these articles may be used as general introductions to the subject:
