Complete holomorphic vector fields and their singular points

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Recap

A holomorphic vector field $X$ on a complex manifold induces a holomorphic foliation (away from the zeros of $X$). $X$ gives each leaf a time form $dT$ — $dT(X) \equiv 1$ — and a translation structure.
Complete and semicomplete vector fields

Complete vector field
Each one-dimensional orbit is of the form $\mathbb{C}/G$ ($G \subset \mathbb{C}$, discrete).

Semicomplete vector field
- For each leaf $L$ there exists $\Omega_L \subset \mathbb{C}$ and a Galois covering map $\phi : \Omega_L \to L$ whose deck transformations are translations.
- For each leaf $L$, each open path $\gamma : [0,1] \to L$, $\int_\gamma dT_L \neq 0$. 
Some facts

- Complete vector fields are semicomplete
- The restriction of a semicomplete vector field is still semicomplete
- It makes sense to speak about germs of semicomplete vector fields
- Germs of semicomplete vector fields give the local models for complete vector fields.
Part III

Non isolated singularities
Vector fields vs. foliations

\[ X = f(x, y) \left( A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} \right), \quad (A, B) = 1 \]

\( f \) defines a curve of singularities (non-isolated singularities).
Affine structures on curves

Affine group

$$\text{Aff}(\mathbb{C}) = \{ z \mapsto az + b \}$$

Affine structure on a curve $L$:
- atlas for $L$ with changes of coordinates in $\text{Aff}(\mathbb{C})$;
- cover $\{ U_i \}$, $L = \bigcup U_i$, $X_i$ vector field on $U_i$ without zeros, $X_i = c_{ij} X_j$, $c_{ij}$ constant; or
- a vector field $X$ in $\tilde{L}$ such that deck transformations act by multiplying $X$ by constants.

Example
Translation structures!
Renormalization and the limit affine structure

The asymptotic affine structure

The affine classes of the translation structures on the leafs of

\[ X = h(x, y)x^n \frac{\partial}{\partial y} \]

with \( h(0) \neq 0 \), extend to \( x = 0 \) in a unique way.

First integrals of \( X \): \( f(x) \).

If \( X' = f(x)x^n \partial/\partial y \) does not vanish at \( x = 0 \) then \( f = g(x)x^{-n} \)
with \( g(0) \neq 0 \):

\[ X'|_{x=0} = g(0) \frac{\partial}{\partial y}. \]

The restrictions of all these vector fields differ by a constant: there is a well-defined affine structure!
Uniformizable affine structures

The affine structure is **uniformizable** if it satisfies one of the following two equivalent properties:

- $L$ is the quotient of $\Omega_L \subset \mathbb{C}$ under a subgroup of $\text{Aff}(\mathbb{C})$.
- for every open path $\gamma : [0, 1] \to L$, the development of the affine structure along $\gamma$ maps the endpoints to different points.

If the affine structure is the translation structure induced by a vector field, this is the same as semicompleteness.
If $X$ is semicomplete vector field on $M$, $C$ a component of the curve zeros of $X$ invariant by the foliation, then the affine structure on $C$ is uniformizable.
Examples

1. $x^2 \partial/\partial x + y(nx - (n + 1)y)\partial/\partial y$, $n \in \mathbb{Z}$, $n \geq 0$,
2. $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$
3. $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$
4. $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$
Uniformizable affine structures (w/singularities on compact curves)

Are:

- Elliptic curves $\mathbb{C}/\Lambda$ and their quotients,
- $\mathbb{CP}^1$ and the quotients by $z \mapsto \omega z$, $\omega^n = 1$,
- $\mathbb{CP}^1$ with the vector field $z \partial / \partial z$, and its quotient under $z \mapsto 1/z$.
- Elliptic curves, quotients of $\mathbb{C}^*$ by groups within \{\(z \mapsto \alpha z\}\}

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Singularities of complete vector fields
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