ArXiv 1709.06850
1808.02201 (w/ P. Cascini)
1908.05037 (w/ R. Svaldi)

Goal: Produce minimal models of foliations

MMP for surface
foliations

Bruschke, Mukai, McOw
Thm X proj. 3-fold \( \Gamma \) contains 1 foliation

simple singularities. Then \( \exists \) an MMP

for \( \Gamma \)

\( \mathcal{X} \to \mathcal{X}' \) of birational contraction

only contracts curves tangent to \( \Gamma \)

for \( \mathcal{X}' \)

1. \( K_{\mathcal{X}'} \) is nef.

2. \( \exists f : \mathcal{X} \to Z - K_{\mathcal{X}} \) is morphic

and fiber 1 of \( f \) gets tangent to \( \Gamma \)

Remark can prove this for \( s = 1 \)-dim pairs
Today: Cone theorem

Thm (Cone + contraction theorem)

Let $X$ be a 3-fold $(F, \Delta)$ be a log canonical pair.

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_F + \Delta \geq 0} + \sum \text{IR}_{\geq 0} [E_i]$$

$E_i$: rational curve tangent to $F$

$0 \leq - (K_F + \Delta \cdot E_i) \leq 6$

Moreover, if $(F, \Delta)$ canonical then $\exists$

a contraction morphism associated to $\text{IR} \geq 0 [E_i]$, for all $i$. 
\(r\)-exceptional divisor

\(q\) canonical singularity provided 
\(a_i \geq 0\) \(V_i, T_i\).

\(q\) log canonical singularity provided 
\(a_i \geq \begin{cases} 0 & \text{if } E_i \text{ is invariant} \\ -1 & \text{if } E_i \text{ is not invariant} \end{cases}\)

Simple singularity (corank 1)

Let \(f\) be a corank 1 foliation
on a form $\omega \in \Omega^1$.

$F$ has simple singularities provided formally about $0$ $F$ is generated by a 1-form of the following form

i) $\sum_{i=1}^5 \lambda_i \frac{dx_i}{x_i}$ s.t. $\sum_{i=1}^5 \lambda_i = 0$ \iff $\psi_{i=0, (i \neq 0)}$

ii) $\sum_{i=1}^5 \pi_i \frac{Ax_i}{x_i} + \psi(x_1, ..., x_5) \sum_{i=2}^5 \frac{x_i dx_i}{x_i}$ s.t. $\sum_{i=1}^5 \lambda_i = 0$ \iff $\psi_{i=0, (i \neq 0)}$
Remark \( \prod_{i=1}^{n} x_i = 0 \) only invariant (formal) hypersurface

Warning! Not all the \( X_i \) are convergent!

Remark: \( n \leq 3 \) we can always resolve a singular to an \( n/1 \) simple sing

(C. Seidenberg, Carol)

In higher dimensions....?
**Bend + Break** (Keel–Matsuki–McKernan, Miyaoka).

$X$ p.r. variety, normal

$\mathcal{F}$ be a foliation

$D_0 \cdots D_n$, $n = \dim X$ be nef & Cartier divisor.

$D_1 \cdots D_n = 0$

$-K_f \cdot D_2 \cdots D_n > 0$

Then, $X$ is covered by ratl divs $\Sigma$

$-K_f \cdot \Sigma > 0$

target to $\mathcal{F}$

$D_i \cdot \Sigma = 0$. \[\square\]
Adjunction

Let \( D \rightarrow X \) be a divisor, \( T \) foliation
compute \((K_7 + \epsilon(CDD))|_D\) in an "intelligent" way.

\[\epsilon(CD) = \begin{cases} 
0 & \text{if } D \text{ is invariant} \\
1 & \text{if } D \text{ is not invariant}
\end{cases}\]

\[\epsilon(CD) = 0\]

By definition we have a morphism

\( T_{Y \to D} \to T_D \)
$(t + t_D)^* \mathcal{T}_D^* = \mathcal{T}_D$

giving an induced foliation on $D$, call it $\mathcal{F}_D$.

Fact: $K_{t + t_D} = K_{t_D} + \Delta_D \quad \Delta_D \geq 0$

($\Delta_D$ is supported on $\text{sing}(T) \cap D$)

Remark: special case when $\text{coker } T = 1$.

In this case $K_{t + t_D} = K_D$.

Let $S_j$ be any collection of $T$-invariant divisors (possibly even formal).
\[(K_x + D + \varepsilon S_x) \mid_D = K_D + \Theta_D\]

\[\Theta_D > 0 \quad (\text{classical adjunction formula})\]

\[\overline{\Delta_D > \Theta_D} \quad (\text{will be important later})\]

\[\mathbb{E}(D) = 1\]

\[N^* \mid_D \leq -\mathcal{R}_x^1\]

\[i \cdot (\psi) \quad \text{defines a foliation} \quad \mathcal{A}_D.\]

\[\text{calc with foliation} \quad \psi_D.\]

\[(K_x + D) \mid_D = K_D + \Theta_D \quad \Delta_D > 0 \quad \text{is canonically defined fibres.}\]
Remark. \((T,D)\) log canonical then \((T_D, \Delta_D)\) is log canonical.

\(r(T) = 1\) \((T,D)\) is log canonical \((K_T + D)\) \(\cong 0\)
\((K_T = 0\) for the foliation by points).

Proof of cone theorem: \(\Delta = 0\)

Goal: show \(R\) is spanned by \(\mathfrak{m}\)
curve.

(i) $H_R^3 = 0$ \text{(not bij)}

(ii) $H_R^3 > 0$ \text{(bij)}

(iii) apply bend & break

$D_i = H_R$ \text{ for } i \leq \nu(H_R)

$D_i = A$ \text{ for } 3 \leq i > \nu(H_R)$

produce result $\sum \Gamma_i \equiv \sum \nu_i H_R = 0 \Rightarrow \Gamma \in \mathbb{R}$. \checkmark

(iii) proceed by induction on dimension

$H_R \nabla G + E \nabla \text{G anhG} \in \mathbb{R}$.
In particular $\exists E_0 \subseteq \text{Supp}(E) \subseteq E_0 \cdot R < 0$.

$R \leq \text{im} (\overline{\text{NE}} (\text{NE}_E) \rightarrow \overline{\text{NE}} (\times 1))$

Want to restrict to $E_0$ and apply the cone theorem in $\dim = 2$

(Boghrosar - M. Chilman)

(i) $E_0$ is invariant

(ii) $E_0$ is not invariant

Focus (iii) (i) is similar

$(K_f + E_0) \cdot R < 0$

$(K_f + E_0)|_{E_0} = K_f + E_0$ by adjunction
\[ R_0 \in \mathcal{N}E \left( E_0 \right) \text{ s.t. } R_0 \text{ is stably and } \left( K_{E_0} + A \right) \text{-negative and s.t. } R_0 \text{ maps to } R \text{ under the inclusion.} \]

Apply the cone theorem for surface solutions to see that \( R_0 \) is spanned by a cut-off curve and \( S \subset R \).