Codimension one foliation with pseudo-effective conormal bundle (PART 2)

Frédéric Touzet

Université de Rennes 1

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Recollections (from PART 1)

We are dealing with codimension one distribution \( D \) defined on a
\( n \)-dimensional compact Kähler manifold \( X \) by \( \text{Ker}(\Omega) \),

\[
\Omega \in H^0(X, \Omega^1_X \otimes L^*)
\]

where \( L \in \text{Pic}(X) \) is pseudo-effective (psef).

By Demailly’s integrability criterion, \( D \) is integrable.

One will set \( \mathcal{F} := D \).
We are dealing with codimension one distribution $\mathcal{D}$ defined on a $n$-dimensional compact Kähler manifold $X$ by $\text{Ker}(\Omega)$,

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By Demailly’s integrability criterion, $\mathcal{D}$ is integrable.

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Example: $\Omega \in H^0(X, \Omega^1_X \otimes \mathcal{O}(-D))$, $D$ effective. $\Omega$ can be viewed as as a holomorphic one form. Here, $L = D$ is psef and can be represented by $[D]$, the integration current of $(n-1, n-1)$ forms on $D$.

In this case, integrability simply follows from $d$-closedness of $\Omega$. 
Foliations defined by holomorphic one forms

We briefly recall the construction of the Albanese torus, $Alb(X)$ associated to $X$ compact Kähler.
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We have natural maps

$$\Omega^1_X(X) \to H^1(X, \mathbb{C})$$

$$H_1(X, \mathbb{Z})/tor \quad \to \quad \Omega^1_X(X)^*$$

$$\gamma \quad \to \quad \Omega \to \int_\gamma \Omega$$

Set $\Gamma = \text{Im}(H_1(X, \mathbb{Z})/tor) \subset \Omega^1_X(X)^*$. It turns out that $\Gamma$ is a cocompact lattice in $\Omega^1_X(X)^*$, so that the quotient

$$Alb(X) := \Omega^1_X(X)^*/\Gamma$$

is a compact torus and called the *Albanese torus* of $X$. 
Fixed once for all a point $x_0$. Integration of forms on paths starting at $x_0$ and ending at $x \in X$ gives a morphism

$$alb_X : X \rightarrow Alb(X).$$

By construction, $alb_X^* : \Omega^1(Alb(X)) \rightarrow \Omega^1_X(X)$ is an isomorphism. In particular, if $\mathcal{F} = Ker(\Omega)$, $\Omega \in \Omega^1_X(X)$, then $\mathcal{F} = alb_X^*(\mathcal{F}_{lin})$ where $\mathcal{F}_{lin}$ is a linear foliation on the torus, i.e given by translation of a codimension one subgroup.
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We would like now to enumerate some criterias which guarantee that $\mathcal{F}$ is defined by a fibration $X \to C$ over a curve $C$.

**Theorem (Castelnuovo-De Franchis)**

Let $\Omega_1, \Omega_2 \in \Omega^1_X(X)$ (compact Kähler), $\Omega_1 \wedge \Omega_2 = 0$. Consider the foliation $\mathcal{F}$ defined by $\text{Ker}(\Omega_1) = \text{Ker}(\Omega_2)$. Then $\mathcal{F}$ is tangent to a fibration $f : X \to C$, where $C$ is a curve of genus $g(C) \geq 2$. 
Proof.

By assumptions, there exists a non constant meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ such that $\Omega_1 = f \Omega_2$. The $d$-closedness of $\Omega_i$ then implies that $\Omega_1 \wedge df = \Omega_2 \wedge df = 0$. Moreover, one can easily check that $f$ has no indeterminacy point (i.e is a genuine holomorphic map) and up to taking Stein factorization, one can suppose that $f : X \to C$ has connected fibers. In particular, $\Omega_1, \Omega_2 \in f^*(\Omega^1_C(C))$ and consequently $g(C) \geq 2$. \qed
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Remark

- **Assumptions of the Theorem** $\Leftrightarrow h^0(N^*F) \geq 2$.
- **If** $\Omega_i \in \Omega^1_X(X), i = 1, 2, 3$ **are three non trivial section of** $N^*F$ **and** $f_1, f_2$ **are the meromorphic function defined by** $\Omega_i = f_i \Omega_{i+1}, i = 1, 2$ **then** $df_1 \wedge df_2 = 0$, **i.e. the forms** $df_1 \wedge df_2$ **are meromorphically independant or equivalently** $\text{trdeg}_C(\mathbb{C}(f_1, f_2)) \leq 1$. 
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- If $\Omega_i \in \Omega^1_X(X)$, $i = 1, 2, 3$ are three non trivial section of $N^*_F$ and $f_1, f_2$ are the meromorphic function defined by $\Omega_i = f_i \Omega_{i+1}, i = 1, 2$ then $df_1 \wedge df_2 = 0$, i.e the forms $df_1 \wedge df_2$ are meromorphically independant or equivalently $\text{trdeg}_\mathbb{C}(\mathbb{C}(f_1, f_2)) \leq 1$.

Actually, one has a more general statement involving **Kodaira dimension**.
Definition
Let $L \in \text{Pic}(X)$ ($X$ compact Kähler). The kodaira dimension of $L$, denoted by $\kappa(L)$ is defined as follows:
Consider the graded ring

$$R(L) := \bigoplus_{m=0}^{\infty} H^0(X, L^\otimes m)$$

and the homogeneous field of fractions

$$Q(L) = \left\{ \frac{l_i}{l_j} \mid l_i, l_j \in H^0(X, L^\otimes m), m \geq 0 \right\}.$$ 

One then sets $\kappa(L) = \text{trdeg}_C Q(X, L)$ if there exists $m > 0$ such that $h^0(L^\otimes m) \neq 0$ and $\kappa(L) = -\infty$ otherwise. Note that $\kappa(L) \leq n := \dim(X)$ and that equality holds if $L$ is ample.
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Theorem (Bogomolov)
Let $X$ be a compact Kähler manifold and a non trivial twisted form $\Omega \in H^0(X, \Omega^1_X \otimes L^*)$, then $\kappa(L) \leq 1$ and when equality holds, $\text{Ker}(\Omega)$ is involutive and tangent to a fibration $X \to \mathbb{C}$. 

Sketch of proof.

If $\kappa(N^*_F) \geq 1$, one comes back to the situation depicted in Castelnuovo-De Franchis Theorem, up to taking some ramified cover, . $\kappa(N^*_F) > 1$ does not occur by the second point of the previous remark. □
Abundance defect

Remark
\[ \kappa(L) \geq 0 \iff h^0(L^\otimes m) \neq 0 \text{ for some } m \implies L \text{ is psef} \]
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It’s then natural to formulate the following

Question
Does there exists an example of codimension one foliation \( \mathcal{F} \) on \( X \) projective such that \( N^*_\mathcal{F} \) is psef and \( \kappa(N^*_\mathcal{F}) = -\infty \) (non abundance phenomenon)?

Yes!

Consider \( X = D_n/\Gamma \) where \( \Gamma < \text{Aut}_0(D_n) = \text{Aut}(D_n) \) is a torsion free irreducible cocompact lattice. \( X \) is endowed with \( n \) codimension one “tautological” foliations \( F_i \) whose lift on \( D_n \) is defined as the kernel of \( dz_i \), \( i = 1, \ldots, n \). These foliations are minimal (i.e have dense leaves), due to the irreducibility of the lattice.
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Consider \( X = \mathbb{D}^n/\Gamma \) where \( \Gamma < Aut^0(\mathbb{D}^n) = Aut(\mathbb{D})^n \) is a torsion free irreducible cocompact lattice. \( X \) is endowed with \( n \) codimension one "tautological" foliations \( \mathcal{F}_i \) whose lift on \( \mathbb{D}^n \) is defined as the kernel of \( dz_i, i = 1, \ldots, n \). These foliations are minimal (i.e have dense leaves), due to the irreducibility of the lattice.
Take for instance $i = 1$. Remark that $\mathcal{F}_1$ is also characterized as the kernel of the semi-positive $(1, 1)$ form

$$
\eta = i \frac{dz_1 \wedge d\overline{z}_1}{(1 - |z_1|^2)}
$$

which is nothing than that the area form of the Poincaré metric on the disk $\mathbb{D}_{z_1}$.

Note that $\eta$ is well defined on $X$ and induces on $N^*_\mathcal{F}_1$ a metric whose curvature is $\eta$ itself (up to some positive factor): this is the dual meaning of the constant negative curvature of the Poincaré metric.
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**Reformulation:** $\mathcal{F}_1$ is equipped with a transversely hyperbolic structure: for a sufficiently fine open cover $(U_i)$ of $X$, $\mathcal{F}_i$ is defined on each $U_i$ by the levels $\{f_i = c\}$, $f_i : U_i \to \mathbb{D}$ subject on overlaps to the glueing conditions $f_i = \alpha_{ij}(f_j)$, $\alpha_{ij} \in Aut(\mathbb{D})$.

Actually, $f_i = z_1 \circ \pi_i^{-1}$ where $\pi_i^{-1}$ is a local inverse of $\pi : \mathbb{D}^n \to X$. 
Let us justify roughly why \( \kappa(N_{\mathcal{F}_1}^*) = -\infty \), assuming for simplicity that \( h^0(N_{\mathcal{F}_1}^*) \neq 0 \). This means that \( \mathcal{F}_1 \) is also defined as the kernel of a holomorphic (hence closed) one form \( \xi \). On \( U_i \), one can write \( \xi = dF_i \).

One can then check, that the collection of Schwarzian derivative \( \{f_i, F_i\} \) of \( f_i \) with respect to \( F_i \) glue together to produce a non constant meromorphic first integral of \( \mathcal{F}_1 \). This contradicts the density of leaves.
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**Question**

Let $\Omega \in H^0(X, \Omega_X^1 \otimes L)$ whith $c_1(L) = 0$ (i.e, $L$ is a flat line bundle). Does this implies that $L$ is torsion ($\iff \kappa(L) = 0$).
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**Question**

Let $\Omega \in H^0(X, \Omega^1_X \otimes L)$ with $c_1(L) = 0$ (i.e, $L$ is a flat line bundle). Does this implies that $L$ is torsion ($\iff \kappa(L) = 0$).

Obviously, the answer is negative in general e.g $X = C$, where $C$ is a curve of genus $\geq 2$.

However, one has the

**Theorem (T, 2016)**

Let $\Omega$ as above ($X$ compact Kähler)

1. Suppose that $\text{codim}\{\Omega = 0\} \geq 2$; then $L$ is torsion

2. If $L$ is not torsion, the foliation $\mathcal{F}$ defined by $\text{Ker}(\Omega)$ is given by a morphism $X \to C$ onto a curve $C$. 
Remark

The first item can be rephrased as $c_1(N^*_F) = 0 \implies N^*_F$ is torsion, where $F = \text{Ker}(\Omega)$ (abundance phenomenon).
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Question

Is it possible to give a "reasonable" description of codimension one foliation $F$ whose conormal bundle violates the abundance principle:

$$N^*_F \text{ psef and } \kappa (N^*_F) = -\infty$$
Statement of the main Theorem

Theorem
Let $\mathcal{F}$ a codimension one foliation on a projective manifold $X$ whose conormal bundle does not satisfy the abundance principle. Then there exists a morphism

$$
\varphi : X \rightarrow \overline{\mathbb{D}^m/\Gamma}^{BB}
$$

such that $\mathcal{F} = \varphi^* \mathcal{F}_i$ for some $i \in \{1, \ldots, m\}$. 

1. $\Gamma$ is an irreducible lattice of $\text{Aut}(\mathbb{D}^m)$
2. $\text{BB}$ denotes the Baily-Borel compactification, which consists here in adding finitely many points (cusps).
3. $\mathcal{F}_i$ is the $i$th tautological foliation (induced by $dz_i$).
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One explains the notations:

1. $\Gamma$ is an irreducible lattice of $\text{Aut}(\mathbb{D})^m$
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Comments

- There exists a logarithmic version of this Theorem (natural because it holds on smooth projective models of $\mathbb{D}^m/\Gamma^{BB}$)

- The projective variety $\mathbb{D}^m/\Gamma^{BB}$ somehow plays the role of the Albanese torus when $\kappa(N^*_F) \geq 0$
Comments

- There exists a logarithmic version of this Theorem (natural because it holds on smooth projective models of $\mathbb{D}^m/\Gamma^{BB}$)
- The projective variety $\mathbb{D}^m/\Gamma^{BB}$ somehow plays the role of the Albanese torus when $\kappa(N^*_F) \geq 0$

We indicate briefly the ingredients of the Proof (details will be given in the last lecture)

1. One construct a hyperbolic transverse invariant metric with respect to $\mathcal{F}$ well defined on $X - H$ where $H$ is a certain hypersurface $\mathcal{F}$-invariant. This produces a morphism $\pi_1(X - H) \rightarrow Aut(\mathbb{D})$, namely the monodromy representation attached to this hyperbolic transverse structure.

2. To conclude, one exploits fundamental results by Corlette and Simpson about rank 2 representations of quasi-projective groups.